# ALGEBRAIC DESCRIPTION OF HOMOGENEOUS CONES

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ABSTRACT. This paper finishes the author's investigations on homogeneous cones. As a result a classification of homogeneous cones is derived. The most important tool to get insight into the structure of homogeneous cones are $J$-morphisms. Therefore, in this paper we mainly deal with morphisms of homogeneous cones. The main result gives an algebraic description of $J$-morphisms. It includes a description of "Linear imbeddings of self-dual homogeneous cones" and the above mentioned classification of homogeneous cones. In a subsequent paper it will be used to describe homogeneous Siegel domains.

In [5] it has been shown how to construct a homogeneous cone in a unique way out of lower dimensional ones. It is clear that in continuing this process one can construct, in a unique way, each homogeneous cone $K$ inductively using "domains of positivity", [9], as building blocks.

Unfortunately, this procedure does not automatically provide a detailed description of the Lie-algebras Lie Aut$(K, \eta)$ appearing in the considerations of [5]. One explanation for this is that the class of functions $\eta: K \to \mathbf{R}^+$ used in [5] is too large. Starting from this point of view, in his thesis, [3], the author considered a rather small class of functions and for the functions $\eta$ of this class gave a detailed description of both $K$ and Lie Aut$(K, \eta)$. However, an analysis of what was done in [3] shows that the appropriate setting for such a description is a "$J$-morphism of homogeneous cones". Roughly speaking, this is a triple consisting of two cones, $K$ in $V$ and $\hat{K}$ in $\hat{V}$, and a linear mapping $\varphi: V \to \hat{V}$ such that $\hat{K}$ is a self-dual cone, $\varphi(K) \subset \hat{K}$ and the group of first components of $\{(W, \hat{W}) \in \text{Aut } K \times \text{Aut } \hat{K}; \varphi(Wx) = \hat{W}\varphi(x) \text{ for all } x \in V\}$ operates transitively on $K$.

Obviously, this definition generalizes the notion of a "representation" of a homogeneous cone which was first introduced by O. Rothaus. There $\hat{K}$ was chosen as a cone of real, positive-definite symmetric matrices. But it also generalizes the setting used in I. Satake's paper *Linear imbeddings of self-dual homogeneous cones* [12]. Here $K$ was assumed to be self-dual.

In this paper we investigate "$J$-morphisms of homogeneous cones". Here we generalize (and publish) the results of [3], VI, and complete the investiga-

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tions of [4], [5], [6], [7a] and [7b]. We prove a theorem which was only announced in [4] and so get a detailed description of homogeneous cones and their Lie-algebras Lie Aut $K$ = Lie Aut($K$, $\iota(K;\ )$).

Moreover, we build the framework for later investigations on homogeneous Siegel domains. It turns out that the results on "$J$-morphisms" can be used to elucidate the structure of those domains and their infinitesimal automorphisms. We shall treat this subject in a subsequent paper.

This paper is organized in 8 sections. The first one contains some basic definitions and remarks. In the second one, general properties of canonical groups are investigated. In §3 a useful description of the Lie-algebras of some trigonalizable groups is given. A reduction of an arbitrary morphism of homogeneous cones to "lower dimensional" ones is derived in §4, so providing an important tool for many inductive steps of the following sections. Beginning in §5 we specialize to "$J$-morphisms". Some homomorphism properties of a $J$-morphism are proved in this section. In §6 we define a $q$-$\Re$-decomposition (see [4]) from a $J$-morphism. Using these preparations we describe in §7 the Lie-algebras Lie Aut($K$, $\gamma$) and Lie Aut$_e$($K$, $\gamma$). These results are analogous to some statements in [15]. The final part, §8, characterizes the $J$-morphisms in algebraic terms. As a result we obtain a sort of classification of homogeneous cones.

The author thanks the referee for proposing to "give a clear definition of all major notations used in the paper, not just referring to earlier papers" and to "add a list of frequently used notations . . . ".

**1. Morphisms of homogeneous cones.** We start with recalling some definitions which form the building blocks for the notations which are introduced later on.

An open convex subset of some finite-dimensional vectorspace $V$ over **R** (the real numbers) which contains with $x$ also $\alpha x$, $\alpha > 0$, but does not contain all of a straight line is called a *regular cone in $V$*.

In what follows we will often deal with the class $\mathcal{F}$, which, by definition, consists of the class of triples $\langle K, \eta, e \rangle$, where $K$ is a regular cone in a finite-dimensional **R**-vectorspace $V$, $e$ is a point of $K$ and $\eta$ is a positive real mapping $\eta: K \to \mathbf{R}^+$ such that

($\mathcal{F}$.1) $\eta$ is infinitely differentiable,

($\mathcal{F}$.2) $\eta$ is homogeneous (there exists $k \in \mathbf{R}$ such that $\eta(\tau x) = \tau^k \eta(x)$ for all $\tau > 0$, $x \in K$),

($\mathcal{F}$.3) the bilinear mapping $(u, v) \mapsto \Delta_x^u \Delta_x^v \log \eta(x)$ is positive-definite, for all $x \in K$ (here $\Delta_x^u$ means differentiation at $x$ in direction $u$),

($\mathcal{F}$.4) for every sequence $x_n$ of $K$ that converges to a boundary point of $K$, the sequence $\eta(x_n)$ converges to $+\infty$,

($\mathscr{F}$.5) the group $\text{Aut}(K, \eta) := \{ W \in \text{GL } V; \; WK = K,$ there exists a $\alpha(W)$
$> 0$ such that $\eta(Wx) = \alpha(W)\eta(x)$ for all $x \in K \}$ operates transitively on $K$.

As in [5] we write $D = \langle K_D, \eta_D, e_D \rangle$ for a triple $D$ of $\mathscr{F}$. Here, by
definition, $K_D$ is an open subset of a finite-dimensional $\mathbf{R}$-vectorspace which
we denote by $V_D$.

For a triple $D = \langle K, \eta, e \rangle$ we define (see also [4], [5])

$$\sigma_D(u, v) := \Delta_x^u \Delta_x^v \log \eta(x)|_{x=e}, \qquad u, v \in V_D, \tag{1.1}$$

$$\sigma_D(h_D(x), u) := -\Delta_x^u \log \eta(x), \qquad u \in V_D, x \in K, \tag{1.2}$$

$$\sigma_D(H_D(x)u, v) := \Delta_x^u \Delta_x^v \log \eta(x), \qquad u, v \in V_D, x \in K, \tag{1.3}$$

$$\sigma_D(uv, w) := -\Delta_x^u \Delta_x^v \Delta_x^w \log \eta(x)|_{x=e}, \qquad u, v, w \in V_D. \tag{1.4}$$

It is well known (and easy to check) that by (1.4) we get an algebra $\mathfrak{A}_D$ which
is defined by the product $(u, v) \mapsto uv$. The algebra $\mathfrak{A}_D$ is commutative and
has $e$ as unit. Further, it is known, [7a, §4], that $h_D$ maps $K$ onto the $\sigma_D$-*dual
cone* $K^{\sigma_D} := \{ y \in V_D; \sigma_D(x, y) > 0$ for all $x \neq 0$ which lie in the closure of
$K$ in $V_D \}$. Moreover, by ($\mathscr{F}$.5) and [6, I, Satz 3.3] we know that $h_D$ and $H_D$ are
rational functions. (As usual a map $f: V \to V'$ is called (a *polynomial*) *rational*
if the components of $f(x)$ with respect to a base of $V'$ are (polynomial)
rational functions of the components of $x$ with respect to a base of $V$.)

We are going to define what we call a morphism of homogeneous cones.

A triple $M = (F, \varphi, \hat{F})$ is called a *morphism of homogeneous cones* iff

(M.1) *$F$ and $\hat{F}$ are triples of $\mathscr{F}$*;

(M.2) *$\varphi: V_F \to V_{\hat{F}}$ is a linear mapping*;

(M.3) *$\varphi(K_F) \subset K_{\hat{F}}$ and $\varphi(e_F) = e_{\hat{F}}$*;

(M.4) *there exists a closed subgroup $\Gamma$ of $\text{Aut}(K_F, \eta_F)$ such that*

   (a) *$\Gamma$ operates transitively on $K_F$,*

   (b) *for all $W \in \Gamma$ there exists a $\hat{W} \in \text{Aut}(K_{\hat{F}}, \eta_{\hat{F}})$ such that $\varphi(Wx) = \hat{W}\varphi(x)$ holds for all $x \in V_F$.*

Let $M = (F, \varphi, \hat{F})$ be a morphism of homogeneous cones. We put

$$\gamma_K: K_F \to \mathbf{R}^+, \quad \gamma_M(x) := \eta_F(x)\eta_{\hat{F}}(\varphi(x)), \qquad x \in K_F, \tag{1.5}$$

$$G(M) := \langle K_F, \gamma_M, e_F \rangle. \tag{1.6}$$

The following lemma states that $G(M)$ is again in $\mathscr{F}$.

LEMMA 1.1. *Let $M$ be a morphism of homogeneous cones. Then $G(M)$ is a
triple of $\mathscr{F}$.*

PROOF. It is easy to see that ($\mathscr{F}$.1), ($\mathscr{F}$.2) and ($\mathscr{F}$.3) are satisfied. Let
$(x_n; n \in \mathbf{N})$ be a sequence of points of $K = K_F$ which converges to a
boundary point of $K$. Then by ($\mathscr{F}$.4) applied to $\hat{\eta}$ we get a constant $C > 0$
such that $\hat{\eta}(\varphi(x_n)) > C > 0$ for all $x_n$. On the other hand $\eta(x_n)$ converges to
$+ \infty$. So $\gamma_M(x_n)$ converges to $+\infty$. Finally, let $\Gamma$ be as in (M.4); then

$\Gamma \subset \text{Aut}(K, \gamma_M)$ and $\text{Aut}(K, \gamma_M)$ operates transitively on $K$.

REMARK 1.2. *In what follows we shall mainly deal with a triple of type $G(M)$. As far as possible we drop the indices $F$ and $G(M)$ and put a "^" on the top instead of writing $\hat{F}$ as a subscript*:

$$K = K_{G(M)} = K_F, \quad e = e_{G(M)} = e_F, \ldots, \hat{K} = K_{\hat{F}}, \quad \hat{e} = e_{\hat{F}}, \ldots .$$

*Further, we write $\gamma = \gamma_M = \gamma_{G(M)}$, $\eta = \eta_F$ and $\hat{\eta} = \eta_{\hat{F}}$. Since we mainly deal with the algebra $\mathfrak{A}_{G(M)}$, we put $\mathfrak{A} := \mathfrak{A}_{G(M)}$. Where the algebra $\mathfrak{A}_F$ appears it will always be clearly distinguishable from $\mathfrak{A}$. Further, we put $\hat{\mathfrak{A}} = \mathfrak{A}_{\hat{F}}$ and denote the left multiplications of $\mathfrak{A}$ (resp. $\hat{\mathfrak{A}}$) by $A(x)$ (resp. $\hat{A}(\hat{x})$).*

**2. Some groups.** Let $M = (F, \varphi, \hat{F})$ be a morphism of homogeneous cones. As $F$ and $\hat{F}$ are triples of $\mathscr{F}$ we know–as remarked in §1–that $h = h_F$ (resp. $\hat{h} = h_{\hat{F}}$) is a rational function in $V = V_F$ (resp. $\hat{V} = V_{\hat{F}}$). Hence there exist polynomials $\pi$, $\nu$ such that $h(x) = [\nu(x)]^{-1}\pi(x)$ for all $x \in V$, $\nu(x) \neq 0$. We put $\Gamma(h) := \{ W \in \text{Gl } V; \; \nu(x)\pi(Wx) = \nu(Wx)\pi(x) \text{ for all } x \in V_F \}$. Analogously we define $\Gamma(\hat{h})$. It is easy to verify that $\Gamma(h)$ resp. $\Gamma(\hat{h})$ equals the group $\Gamma(\eta')$ resp. $\Gamma(\hat{\eta}')$ as defined in [6, I, §1]. We may apply [6, I, Lemma 1.5 and I, Satz 1.8] to both groups. Hence we know that $\text{Aut}(K, \eta)$ is a closed subgroup of the algebraic group $\Gamma(h)$. It has finite index; so their Lie-algebras coincide. The analogous statement holds for $\Gamma(\hat{h})$.

We put

$$\Pi_M := \{(W, \hat{W}) \in \Gamma(\eta') \times \Gamma(\hat{\eta}'); \; \varphi(Wx) = \hat{W}\varphi(x) \text{ for all } x \in V \}, \quad (2.1)$$

$$\Pi_M^a := (\text{Aut}(K, \eta) \times \text{Aut}(\hat{K}, \hat{\eta})) \cap \Pi_M. \quad (2.2)$$

It is easy to see that in this definition we could have replaced $\eta$ by $\gamma$ without changing the resulting sets $\Pi_M$ and $\Pi_M^a$.

In what follows we denote the Lie-algebra of a Lie-group $\Gamma$ by Lie $\Gamma$.

LEMMA 2.1. *Let $M = (F, \varphi, \hat{F})$ be a morphism of homogeneous cones.*
(a) *$\Pi_M$ is a linear algebraic subgroup of $\text{GL } V \times \text{GL } \hat{V}$.*
(b) *$\Pi_M^a$ is a closed subgroup of $\Pi_M$.*
(c) *Lie $\Pi_M = \{(T, \hat{T}) \in \text{Lie Aut}(K, \eta) \times \text{Lie Aut}(\hat{K}, \hat{\eta}); \; \varphi(Tx) = \hat{T}\varphi(x)$ for all $x \in V\}$.*
(d) *Lie $\Pi_M = $ Lie $\Pi_M^a$.*

PROOF. The parts (a), (b) and (c) are obvious. Part (d) follows easily from the abovementioned fact that $\text{Aut}(K, \eta)$ resp. $\text{Aut}(\hat{K}, \hat{\eta})$ has the same Lie-algebra as $\Gamma(h)$ resp. $\Gamma(\hat{h})$.

In what follows we denote the projection of $V \times \hat{V}$ onto $V$ resp. $\hat{V}$ by a point "˙" resp. double point "¨" put over the projected element; e.g., for $a \in V \times \hat{V}$ we write $a = (\dot{a}, \ddot{a})$. In the same manner we deal with the corresponding projections of $\text{End } V \times \text{End } \hat{V}$ onto $\text{End } V$ resp. $\text{End } \hat{V}$.

Let now $\Gamma$ be a Lie-group; then $\Gamma^0$ denotes the connected component of $\Gamma$ that contains the unit element of $\Gamma$.

Finally, we call a set $\Gamma$ of endomorphisms of a vector space $R$ *trigonalizable* iff there exists a basis of $R$ such that the elements of $\Gamma$ are upper triangular matrices with respect to this basis.

REMARK 2.2. (a) From [**14**, I, §6] we recall that a connected, trigonalizable subgroup of $\text{Aut}(K, \eta)$ and a compact subgroup of $\text{Aut}(K, \eta)$ have a trivial intersection. Further, it is known [**14**, I, §6] that every compact subgroup of $\text{Aut}(K, \eta)$ has a fixed point in $K$ and that every isotropy subgroup of $\text{Aut}(K, \eta)$ is compact.

(b) From these remarks one easily obtains:

Let $\Psi_1 \subset \Psi_2$ be connected, trigonalizable subgroups of $\text{Aut}(K, \eta)$, and assume that $\Psi_1$ operates transitively on $K$. Then $\Psi_1 = \Psi_2$ and $\Psi_1$ is a closed subgroup of $\text{Aut}(K, \eta)$.

LEMMA 2.3. *Let $M$ be a morphism of homogeneous cones.*

(a) *Let $\Phi$ be a connected, trigonalizable Lie-subgroup of $\Pi_M$ such that $\dot\Phi$ operates transitively on $K$. Then $\Phi$ and $\dot\Phi$ are closed, and we have $\widehat{\text{Lie}\ \Phi} = \text{Lie}\ \dot\Phi$.*

(b) *$\dot\Pi_M^a$ contains a connected, trigonalizable subgroup which operates transitively on $K$.*

(c) *Let $\Psi$ be a connected, trigonalizable Lie-subgroup of $\dot\Pi_M^a$ which operates transitively on $K$. Then there exists a connected, trigonalizable Lie-subgroup $\Phi$ of $\Pi_M^a$ such that $\dot\Phi = \Psi$ and $\text{Lie}\ \dot\Phi = \widehat{\text{Lie}\ \Phi} = \text{Lie}\ \Psi$.*

PROOF. (a) $\Phi$ is closed by [**13**, Proposition 1]. As $\Phi$ is connected we have $\Phi \subset \Pi_M^a$ by Lemma 2.1. Hence $\dot\Phi \subset \text{Aut}(K, \eta)$ is connected, trigonalizable and operates transitively on $K$. So Remark 2.2 applies and $\dot\Phi$ is closed. Finally, we note that $\Phi \to \dot\Phi$, $Q \mapsto \dot Q$ is a surjective homomorphism of Lie-groups. The last statement follows.

(b) From (M.4) we conclude that $\Pi_M^a \times K \to K$, $(Q, x) \mapsto \dot Q x$ defines a transitive action of $\Pi_M^a$ on $K$. By [**8**, II, §4] $(\Pi_M^a)^0$ operates transitively on $K$. From Lemma 2.1 we get $(\Pi_M^a)^0 = \Pi_M^0$. On the other hand, we can write $\Pi_M^0 = \Phi_t\Phi_c$ with a connected, trigonalizable subgroup $\Phi_t$ and a connected, compact subgroup $\Phi_c$ by [**14**, I, §6]. Putting all this together, we see that $\dot\Phi_t\dot\Phi_c$ operates transitively on $K$. As $\dot\Phi_t$, $\dot\Phi_c \subset \text{Aut}(K, \eta)$ and $\dot\Phi_c$ is compact, there exists a fixed point in $K$ for $\dot\Phi_c$. Hence $\dot\Phi_t$ satisfies the requirements of the assertion.

(c) Let $\Xi$ be the smallest algebraic subgroup of $\Pi_M$ that contains $\{Q \in \Pi_M; \dot Q \in \Psi\}$. Then $\dot\Xi$ is trigonalizable and Remark 2.2 together with Lie $\Gamma(h) = \text{Lie Aut}(K, \eta)$, [**6**, Satz 1.8], shows $\Psi = (\dot\Xi)^0$. Now let $\Xi^a := \Xi \cap \Pi_M^a$.

Then $\Psi \subset \dot{\Xi}^a$ and $\Xi^a$ operates transitively on $K$ by $\Xi^a \times K \to K$, $(Q, x) \mapsto \dot{Q}x$; so do $(\Xi^a)^0$ and $\widehat{(\Xi^a)^0}$. It follows that $\widehat{(\Xi^a)^0} \subset \widehat{\Xi^0} \subset (\dot{\Xi})^0 = \Psi$. Again Remark 2.2 implies that all inclusions are equalities showing $\Psi = \widehat{\Xi^0}$. Write now $\Xi^0 = \Phi\Phi_c$ as the product of a connected, trigonalizable group $\Phi$ and a connected, compact group $\Phi_c$ by [14, I, §6]. Then we have $\Psi = \widehat{\Xi^0} = \dot{\Phi}\dot{\Phi}_c$. But $\dot{\Phi}_c$ is connected, compact and trigonalizable, hence $\dot{\Phi}_c = \{\text{Id}\}$, and the assertion follows.

COROLLARY 2.4. *Let $M$ be a morphism of homogeneous cones. Then there exists a connected, trigonalizable Lie-subgroup $\Phi$ of $\Pi_M^a$ such that $\dot{\Phi}$ operates transitively on $K$.*

Finally, we note a useful result.

LEMMA 2.5. *Let $M$ be a morphism of homogeneous cones and $\Phi$ a connected, trigonalizable subgroup of $\Pi_M^a$. Then there exists a homomorphism $\hat{\ } : \dot{\Phi} \to \ddot{\Phi}$, $W \mapsto \hat{W}$ such that $\Phi = \{(W, \hat{W}); W \in \dot{\Phi}\}$.*

PROOF. Let $W \in \dot{\Phi}$, then there exists a $Q \in \ddot{\Phi}$ such that $(W, Q) \in \Phi$. It obviously suffices to show that $Q$ is uniquely determined. We therefore have to prove that $(\text{Id}, Q) \in \Phi$ implies $Q = \text{Id}$. But by assumption $\ddot{\Phi}$ is connected, trigonalizable and contained in $\text{Aut}(\hat{K}, \hat{\eta})$ for $M = (F, \varphi, \hat{F})$. Further, $\hat{e} = \varphi(\text{Id } e) = Q\varphi(e) = Q\hat{e}$. By Remark 2.2 we get $Q = \text{Id}$.

**3. The Lie-algebras of trigonalizable subgroups of $\Pi_M$.** In what follows we often use results of [5]. We therefore recall some notation from [5]. Let $R$ be a triple of $\mathcal{F}$ and $\mathfrak{A}_R$ the algebra which is associated to $R$ by (1.4). Denote by $B(x)$, $x \in V_R$, the left multiplications in $\mathfrak{A}_R$. Then we put

$$\mathfrak{X}_R := \{x \in V_R; B(x) \in \text{Lie Aut}(K_R, \eta_R)\}. \tag{3.1}$$

We know (see e.g. [5, Corollary 5.4]) that $\mathfrak{X}_R$ is a formally-real Jordan-algebra. We call elements $f_1, \ldots, f_r \in \mathfrak{X}_R$ a *complete system of orthogonal idempotents* (short: CSI) if $f_i = e_R$, $f_i \neq 0$ and $f_i f_j = \delta_{ij} f_i$ holds. For a CSI $f_1, \ldots, f_r$, by [7b, §3] we may form the *$r$-Peirce-decomposition* $(\mathfrak{A}_{ij}; 1 \leq i, j \leq r)$ of $\mathfrak{A}_R$ *with respect to* $f_1, \ldots, f_r$. Where it is possible we drop "$r$" and just write Peirce-decomposition. The spaces $\mathfrak{A}_{ij}$ are defined by

$$\mathfrak{A}_{ij} := \left\{x \in \mathfrak{A}_R; f_i x = \tfrac{1}{2}x, f_j x = \tfrac{1}{2}x\right\} \quad \text{for } i \neq j,$$
$$\mathfrak{A}_{ii} := \{x \in \mathfrak{A}_R; f_i x = x\}. \tag{3.2}$$

We have the following "multiplication table"

$$\mathfrak{A}_{ij}\mathfrak{A}_{kl} = \{0\} \quad \text{if } \{i, j\} \cap \{k, r\} = \varnothing,$$
$$\mathfrak{A}_{is}\mathfrak{A}_{sl} \subset \mathfrak{A}_{il} \quad \text{if } i \neq l,$$
$$\mathfrak{A}_{is}\mathfrak{A}_{is} \subset \mathfrak{A}_{ii} + \mathfrak{A}_{ss}. \tag{3.3}$$

Further, the "Peirce-spaces" $\mathfrak{A}_{ij}$ are orthogonal by pairs with respect to $\sigma_R$. We remark that the notion of a Peirce-decomposition in the sense of [**7b**, §3] is the same as that of [**4**, §1], 3. If the CSI is of the type $c$, $e_R - c$ we will use the usual notation $\mathfrak{A}_1 = \mathfrak{A}_1(c) = \mathfrak{A}_{11}$, $\mathfrak{A}_{1/2} = \mathfrak{A}_{1/2}(c) = \mathfrak{A}_{12}$, $\mathfrak{A}_0 = \mathfrak{A}_0(c) = \mathfrak{A}_{22}$. If $L(x)$ denotes the left multiplications in $\mathfrak{A}$ then by $L_{1/2}(x_1 + x_0)$ we denote the restriction of $L(x_1 + x_0)$ to $\mathfrak{A}_{1/2}(c)$, $x_i \in \mathfrak{A}_i(c)$.

Besides the notion of a Peirce-decomposition we have to recall the definition of a ctt-group and a "mutant".

A connected, trigonalizable subgroup of $\text{Aut}(K_R, \eta_R)$ that operates transitively on $K_R$ is called a ctt-*group for R*. A Lie-algebra is a ctt-*algebra for R* if it is the Lie-algebra of a ctt-group for $R$.

Finally, we define for a commutative algebra $\mathfrak{B}$ with left multiplications $B(x)$, $x \in \mathfrak{B}$, and product $(x, y) \mapsto xy$, a new algebra for each $b \in \mathfrak{B}$ on the vectorspace $\mathfrak{B}$ by the product $(x, y) \mapsto x(by) + (xb)y - b(xy)$. This new algebra is called "*mutant*" (of $\mathfrak{B}$ with respect to $b$) and is denoted by $\mathfrak{B}_b$. For the left multiplications of $\mathfrak{B}_b$ we write $B_b(x)$, $x \in \mathfrak{B}$. Obviously, we have $B_b(x) = B(bx) + [B(x), B(b)]$.

THEOREM 3.1. *Let $M = (F, \varphi, \hat{F})$ be a morphism of homogeneous cones and $\Phi$ a connected, trigonalizable subgroup of $\Pi_M^a$ such that $\dot{\Phi}$ operates transitively on $K$. Let further $c_1, \ldots, c_r \in \mathfrak{X}_{G(M)}$ be a CSI such that $A(c_i) \in \text{Lie } \dot{\Phi}$, and let $\mathfrak{A} = \bigoplus_{1 \leqslant i \leqslant j \leqslant r} \mathfrak{A}_{ij}$ be the Peirce-decomposition of $\mathfrak{A}$ with respect to $c_1, \ldots, c_r$. Put now $\hat{c}_i := \varphi(c_i)$, $1 \leqslant i \leqslant r$, and*

$$\mathfrak{t}_{ij} := \{(T, \hat{T}) \in \text{Lie } \Phi, Te \in \mathfrak{A}_{ij}\}, \qquad 1 \leqslant i \leqslant j \leqslant r.$$

*Then we have*

(a) *$\hat{c}_1, \ldots, \hat{c}_r \in \hat{\mathfrak{X}}$ form a CSI of $\hat{\mathfrak{A}}$.*

(b) *Lie $\Phi = \bigoplus_{1 \leqslant i \leqslant j \leqslant r} \mathfrak{t}_{ij}$ (direct sum of vector spaces).*

(c) *If $i < j$ and $(T, \hat{T}) \in \mathfrak{t}_{ij}$, then $T = 2A_{c_i}(Tc_i) + 2A_{c_j}(Tc_j)$ and $\hat{T} = 2\hat{A}_{\hat{c}_i}(\hat{T}\hat{c}_i) + 2\hat{A}_{\hat{c}_j}(\hat{T}\hat{c}_j)$.*

(d) *Let $(T, \hat{T}) \in \text{Lie } \Phi$ such that $Te = c_i$; then $T = A(c_i)$ and $\hat{T} = \hat{A}(\hat{c}_i)$.*

PROOF. By assumption $\Phi$ is connected and trigonalizable, and, consequently, so are $\dot{\Phi}$ and $\ddot{\Phi}$. Let now $\hat{\Phi}$ be a maximal connected, trigonalizable subgroup of $\text{Aut}(\hat{K}, \hat{\eta})$ which contains $\ddot{\Phi}$. As $\text{Aut}(\hat{K}, \hat{\eta})$ contains a ctt-group (for definition see above) it follows by [**13**, Theorem 2] that $\hat{\Phi}$ is a ctt-subgroup of $\text{Aut}(\hat{K}, \hat{\eta})$. Hence Lie $\hat{\Phi}$ is a ctt-algebra for $\hat{F}$.

(d) Let $(T, \hat{T}) \in \text{Lie } \Phi$ such that $Te = c_i$. Then $T = A(c_i)$ and $Te = T^2 e$. But by Lemma 2.1(c) we have $\hat{T}^2\hat{e} = \varphi(T^2 e) = \varphi(Te) = \hat{T}\hat{e}$, so that $\hat{T} = \hat{A}(\hat{T}\hat{e}) = \hat{A}(\hat{c}_i)$ by [**5**, Corollary 5.2].

(a) Because of (d) it suffices to show $\hat{c}_i\hat{c}_j = \delta_{ij}\hat{c}_i$ as $\hat{e} = \Sigma'_{i=1}\hat{c}_i$ is obvious. Let $(T, \hat{T}) \in \text{Lie } \Phi$ such that $Te = c_i$; then

$$\hat{c}_i\hat{c}_j = \hat{A}(\hat{c}_i)\varphi(c_j) = \hat{T}\varphi(c_j) = \varphi(Tc_j) = \varphi(c_ic_j) = \delta_{ij}\varphi(c_i) = \delta_{ij}\hat{c}_i.$$

(b) It is clear that Lie $\Phi$ is equal to the sum of the $t_{ij}$. So let $0 = \Sigma_{1 < i < j < r}(T_{ij}, \hat{T}_{ij})$; then $0 = \Sigma T_{ij}e$ and as $T_{ij}e \in \mathfrak{A}_{ij}$ we get $T_{ij}e = 0$. From Remark 2.2 we conclude $T_{ij} = 0$ for $1 < i < j < r$. But now $\hat{T}_{ij} = 0$ by Lemma 2.5.

(c) Let $d_1, \ldots, d_s$ be idempotents of $\mathfrak{X}_{G(M)}$ that satisfy the assertions of [5, Theorem 5.1] with respect to $\mathfrak{A} = \text{Lie }\dot{\Phi}$ (it is clear that Lie $\dot{\Phi}$ is a ctt-algebra for $G(M)$, [5, §5, 1]). By assumption $T = A(c_i) \in \text{Lie }\dot{\Phi}$ and $T^2e = Te$ follows. From [14, Chapter II, Proposition 9] we get a subset $I_i \subset \{1, \ldots, s\}$ such that $c_i = \Sigma_{k \in I_i} d_k$. Obviously, the sets $I_i$ yield a partition of $\{1, \ldots, s\}$. We form the Peirce-decomposition $\mathfrak{A} = \bigoplus_{1 < p < q < s} \mathfrak{A}_{pq}^*$ with respect to $d_1, \ldots, d_s$, and get $\mathfrak{A}_{ij} = \bigoplus \mathfrak{A}_{pq}^*$ where the sum is indexed by $\{(p, q); p < q, p \in I_i, q \in I_j \text{ or } q \in I_i, p \in I_j\}$.

It obviously suffices to prove the assertion for $(T, \hat{T}) \in \text{Lie }\Phi$ which satisfy $Te \in \mathfrak{A}_{pq}^*$. If $p \in I_j$ and $q \in I_i$, then by [5, Theorem 5.1], we have $T = 2A_{d_q}(Te)$. Let $n \in I_j$, $n \neq p$; then $A_{d_n}(Te) = 0$ by [4, Lemma 1.1] since $n \notin \{p, q\}$ and $Te \in \mathfrak{A}_{pq}^*$. It follows $T = 2A_{c_j}(Te)$. But this implies $Te = Tc_j$ and $Tc_i = 0$. Hence $T$ has the required form.

The assertion on $\hat{T}$ is analogously proved by using idempotents $d_1, \ldots, d_s \in \mathfrak{X}_{\hat{F}}$ which are chosen with respect to Lie $\hat{\dot{\Phi}}$.

Finally, we affirm that the situation described in Theorem 3.1 occurs. Using [5, Theorem 5.1] we have

THEOREM 3.2. *Let $M$ be a morphism of homogeneous cones and $\Phi$ a connected, trigonalizable subgroup of $\Pi_M^a$ such that $\dot{\Phi}$ operates transitively on $K$. Then there exists a CSI $d_1, \ldots, d_r \in \mathfrak{X}_{G(M)}$ such that $A(d_i) \in \text{Lie }\dot{\Phi}$ and $\mathfrak{A}_1(d_i) = \mathbf{R}d_i$ for $1 < i < r$.*

**4. Reduction to lower dimension.** In this section we construct new morphisms $M_i$ for each member $M$ of a special class of morphisms of homogeneous cones. These $M_i$ are built up by the constituents which have lower dimension than those of $M$. Before we can start to explain this in detail we have to recall some notations.

First of all, for a regular cone $K$ in $V$ one defines the "*invariant*" $\iota(K; \cdot)$ of $K$ by

$$\iota(K; x) := \int_{K^*} e^{-\lambda(x)} \, d\lambda$$

where $d\lambda$ denotes the Lebesgue-measure on the dual space $V^*$ of $V$ and $K^*$ the *dual cone for K*, $K^* := \{\lambda \in V^*; \lambda(x) > 0 \text{ for all } x \neq 0 \text{ in the closure of } K \text{ in } V\}$. We know $\iota(K; Wx) = |\det W|^{-1}\iota(K; x)$ for all $x \in K$, $W \in \text{Gl } V$, $WK = K$ (see e.g. [7a, §3]). In the case $K = K_R$ where $R$ is a triple of $\mathcal{F}$ we

have $\text{Aut}(K_R, \eta_R) \subset \text{Aut}(K_R; \iota(K_R; \cdot))$. For more information on the function $\iota(K; \cdot)$ we refer to [7a] and [9].

In the remainder of this paper we repeatedly use a construction which has been carried out in [7a, §8]. Therefore we shortly describe again this construction. Let $R = \langle K, \eta, e \rangle$ be a triple of $\mathscr{F}$; then we define $h = h_R$ by (1.2). We know, as remarked in §1, that $h$ maps diffeomorphically $K$ onto $K^\sigma$, $\sigma := \sigma_R$. Hence we may define a function $\tilde{\eta}: K^\sigma \to \mathbf{R}^+$ by $\tilde{\eta}(y) := [\eta(h^{-1}(y))]^{-1}$, $y \in K^\sigma$. We form the triple $\tilde{R} := \langle K^\sigma, \tilde{\eta}, e \rangle$. From [7b, §1, 5] we get that $\tilde{R}$ is a triple of $\mathscr{F}$. More precisely, we have $e_{\tilde{R}} = e_R$, $\sigma_{\tilde{R}} = \sigma_R$, $h_{\tilde{R}} = (h_R)^{-1}$, $\mathfrak{A}_{\tilde{R}} = \mathfrak{A}_R$ and $\text{Lie Aut}(K^\sigma, \tilde{\eta}) = [\text{Lie Aut}(K, \eta)]^\sigma$, $\sigma = \sigma_R$. We shall prove in §7 that for a morphism $M$ of homogeneous cones we get a detailed description of $\text{Lie Aut}(K^\sigma, \eta_{\widetilde{G(M)}})$, $\sigma = \sigma_{G(M)}$. The procedure which we are going to work out in this section is a first step towards such a description. To be able to do this step we have to use the set $\mathfrak{S}_R$, $R$ a triple of $\mathscr{F}$, which was defined in [5, (1.14)]: Let $L(x)$, $x \in V_R$, denote the left multiplications in the algebra $\mathfrak{A}_R$; then we have $\mathfrak{S}_R := \{x \in V_R; L_x(v) \in \text{Lie Aut}(K_R, \eta_R) \text{ for all } v \in V_R\}$. Here we used the left multiplications $L_x(v)$ of the mutant $(\mathfrak{A}_R)_x$. From [5, Theorem 1.2] we know that $\mathfrak{S}_R$ is a formally-real algebra and a subalgebra of $\mathfrak{A}_R$. Hence $\mathfrak{S}_R$ has a unit. We denote this unit by $c_R$. For a morphism $M$ of homogeneous cones we introduce the following convention: *we denote the unit $c_{G(M)}$ of $\mathfrak{S}_{G(M)}$ by $c$ and the unit $c_{\widetilde{G(M)}}$ of $\mathfrak{S}_{\widetilde{G(M)}}$ by $\tilde{c}$.*

This finishes the introduction of the notations which are used in the following.

*In this section let $M = (F, \varphi, \hat{F})$ be a morphism of homogeneous cones that satisfies*

(M.5) $\eta_F = \iota(K_F; )$; $\eta_F(e_F) = 1$.

We first state

(4.1) $(A(\tilde{c}), \hat{A}(\varphi(\tilde{c}))) \in \text{Lie } \Phi$ *for every connected, trigonalizable subgroup of $\Pi_M^a$ for which $\dot{\Phi}$ acts transitively on $K$.*

To prove (4.1) choose $\Phi$ as in the assertion. Then–using the remark before Lemma 2.1 and the fact $\text{Lie Aut}(K, \gamma) = \text{Lie } \Gamma(h_{G(M)})$–we see that $\text{Lie } \dot{\Phi}$ is a ctt-algebra for $G(M)$. Then for $\sigma = \sigma_{G(M)}$ the algebra $\text{Lie } \dot{\Phi}^\sigma$ is a ctt-algebra for $\widetilde{G(M)}$ by [7b, §1], and so contains $A(\tilde{c})$ by [5, Corollary 6.2]. This implies that $A(\tilde{c})$ is an element of $\text{Lie } \dot{\Phi}$ and Theorem 3.1 (d) proves the assertion.

We form the Peirce-decomposition $\mathfrak{A} = \mathfrak{A}_1 + \mathfrak{A}_{1/2} + \mathfrak{A}_0$ of $\mathfrak{A} = \mathfrak{A}_{G(M)}$ with respect to $\tilde{c}$, i.e., $\mathfrak{A}_i = \mathfrak{A}_i(\tilde{c})$. Analogously, we have $\hat{\mathfrak{A}} = \hat{\mathfrak{A}}_1 + \hat{\mathfrak{A}}_{1/2} + \hat{\mathfrak{A}}_0$, $\hat{\mathfrak{A}}_i = \hat{\mathfrak{A}}_i(\varphi(\tilde{c}))$. Further, we put $p_1 := p := \tilde{c}$, $\hat{p}_1 := \hat{p} := \varphi(p) = \varphi(\tilde{c})$ and $p_0 := e - p_1$, $\hat{p}_0 := \hat{e} - \hat{p}_1$. Then we claim

$$\varphi(\mathfrak{A}_i) \subset \mathfrak{A}_i \quad \text{for } i = 0, \tfrac{1}{2}, 1. \tag{4.2}$$

$$\varphi(A_{p_0}(x_{1/2})y) = \hat{A}_{\hat{p}_0}(\varphi(x_{1/2}))\varphi(y) \quad \text{for } x_{1/2} \in \mathfrak{A}_{1/2}, y \in \mathfrak{A}. \tag{4.3}$$

To prove this take $\Phi$ as in Corollary 2.4. Then because of (4.1) we may apply Theorem 3.1(c) and have $T = 2A_{p_1}(Tp_1) + 2A_{p_0}(Tp_0)$ and $\hat{T} = 2\hat{A}_{\hat{p}_1}(\hat{T}\hat{p}_1) + 2\hat{A}_{\hat{p}_0}(\hat{T}\hat{p}_0)$ for $(T, \hat{T}) \in \text{Lie } \Phi$ satisfying $Te \in \mathfrak{A}_{1/2}$. As in the proof for (4.1) we see that Lie $\dot{\Phi}^\sigma$ is a ctt-algebra for $G(M)$, hence $A_{p_1}(Tp_0) \in \text{Lie } \dot{\Phi}^\sigma$ by [5, Theorem 6.1]. Since $T^\sigma \in \text{Lie } \dot{\Phi}^\sigma$ we get $A_{p_0}(Tp_1) \in \text{Lie } \dot{\Phi}^\sigma \subset \text{Lie Aut}(K_{\widetilde{G(M)}}, \tilde{\gamma})$. Therefore, $Tp_1 = 0$ by [5, Theorem 3.3] and the choice of $p_1 = \tilde{c}$. This implies $\hat{T}\hat{p}_1 = 0$ and (4.3) is proved. The inclusions (4.2) are easily seen by using (4.1).

Now choose $\Phi$ as in Corollary 2.4 and $\hat{\dot{\Phi}} \supset \ddot{\Phi}$ as in the proof of Theorem 3.1. Then Lie $\dot{\Phi}$ (resp. Lie $\hat{\dot{\Phi}}$) is a ctt-algebra for $G(M)$ (resp. $\hat{F}$). For $p_1$ (resp. $\hat{p}_1$) we may–because of (4.1)–apply [5, §6, 2]. Hence for $i = 0, 1$ we get cones $K_i$ (resp. $\hat{K}_i$) in $\mathfrak{A}_i$ (resp. $\hat{\mathfrak{A}}_i$) and functions $\gamma_i$ (resp. $\hat{\eta}_i$). We put $\sigma = \sigma_{G(M)}$ and define $P(\mathfrak{A}_{1/2}, \sigma)$ to be the cone of positive definite, selfadjoint (with respect to $\sigma$) endomorphisms of $\mathfrak{A}_{1/2}$. Further, let $\text{Sym}(\mathfrak{A}_{1/2}, \sigma)$ denote the space of selfadjoint endomorphisms of $\mathfrak{A}_{1/2}$ with respect to $\sigma$, then for $i = 0, 1$ we put $\varphi_i \colon \mathfrak{A}_i \to \text{Sym}(\mathfrak{A}_{1/2}, \sigma) \times \hat{\mathfrak{A}}_i$, $x_i \mapsto (A_{1/2}(x_i), \varphi(x_i))$. (Formally the definition depends on $\Phi$, but (4.1) shows that this is actually not the case.)

$$\varphi_i(K_i) \subset P(\mathfrak{A}_{1/2}, \sigma) \times \hat{K}_i \quad \text{for } i = 0, 1. \tag{4.4}$$

$$\varphi_i(p_i) = \left(\tfrac{1}{2}\text{Id}, \hat{p}_i\right) \quad \text{for } i = 0, 1. \tag{4.5}$$

Here (4.5) is trivial and $\varphi(K_i) \subset \hat{K}_i$ follows easily from the definitions. To see $A_{1/2}(K_i) \subset P(\mathfrak{A}_{1/2}, \sigma)$ we first apply [5, Theorem 1.8] to $K^\sigma = K_{\widetilde{G(M)}}$ and get $K^\sigma = \bigcup_{x_{1/2} \in \mathfrak{A}_{1/2}} \exp A_{p_1}(x_{1/2})(Y_1 + Y_0)$ where $Y_i$ is the image of $K^\sigma$ under the projection of $\mathfrak{A} = \mathfrak{A}_1 + \mathfrak{A}_{1/2} + \mathfrak{A}_0$ onto $\mathfrak{A}_i$. As mentioned in §1 (and repeated above) we have $h_{\widetilde{G(M)}}(K^\sigma) = (K^\sigma)^\sigma = K$. Further, we know $h_{\widetilde{G(M)}}(Wy) = W^{\sigma-1}h_{\widetilde{G(M)}}(y)$ for all $y \in K^\sigma$, $W \in \text{Aut}(K^\sigma, \tilde{\gamma})$ by [5, (1.12)] and $\exp A_{p_1}(x_{1/2}) \in \text{Aut}(K^\sigma, \tilde{\gamma})$ by definition of $\mathfrak{S}_{\widetilde{G(M)}}$ and $[A_{p_1}(x_{1/2})]^\sigma = A_{p_0}(x_{1/2})$ by [4, Folgerung 1.2]. Finally, we use part (b) of [5, Lemma 2.2] and get $K = h_{\widetilde{G(M)}}(K^\sigma) = \bigcup_{x_{1/2} \in \mathfrak{A}_{1/2}} \exp A_{p_0}(x_{1/2})(Y_1^\sigma + Y_0^\sigma)$. From this we deduce $Y_i^\sigma = K_i$. (Note that for simplicity of notation we denote the $\sigma_i := \sigma|_{\mathfrak{A}_i \times \mathfrak{A}_i}$–dual cone for $Y_i$ by $Y_i^\sigma$.)

So from [5, Lemma 3.6, Theorem 3.7], we easily conclude $\varphi(K_i) = \varphi(Y_i^\sigma) \subset P(\mathfrak{A}_{1/2}, \sigma)$.

For $i = 0, 1$ put $r_i \colon K_i \to \mathbf{R}^+$, $x_i \mapsto \iota(K_i; x_i)$, $r_i(p_i) = 1$, and $\hat{r}_i \colon P(\mathfrak{A}_{1/2}, \sigma) \times \hat{K}_i \to \mathbf{R}^+$, $(A, \hat{x}_i) \mapsto (\det A)^{-1/2}\hat{\eta}_i(\hat{x}_i)$ where $\hat{\eta}_i$ is the same as in the remarks before (4.4). We further define $R_i := \langle K_i, r_i, p_i \rangle$ and $\hat{R}_i := \langle P(\mathfrak{A}_{1/2}, \sigma) \times \hat{K}_i, \hat{r}_i, (\tfrac{1}{2}\text{Id}, \hat{p}_i) \rangle$ for $i = 0, 1$.

(4.6) $R_i$ and $\hat{R}_i$ are triples of $\mathscr{F}$, $i = 0, 1$.

To prove this we observe that by [5, Corollary 6.5] the functions $\hat{\eta}_i$ satisfy the conditions ($\mathscr{F}$.1) to ($\mathscr{F}$.5). Now the assertion follows easily.

We recall that $\Phi$ was chosen as in Corollary 2.4. For $i \in \{0, 1\}$ let $Q \in \Phi$

be such that $\dot{Q}\mathfrak{A}_j \subset \mathfrak{A}_j$ for $j = 0, \frac{1}{2}, 1$ and $\dot{Q}\mathfrak{A}_{1-i} = 0$. Further, we put $\dot{Q}_{1/2} := \dot{Q}|_{\mathfrak{A}_{1/2}}$ for such a $Q \in \Phi$.

(4.7) $\varphi_i(\dot{Q}x_i) = (\dot{Q}_{1/2}A_{1/2}(x_i)\dot{Q}_{1/2}^\sigma, \ddot{Q}\varphi(x_i))$ *for all* $i \in \{0, 1\}$, $x_i \in \mathfrak{A}_i$, *and all* $Q \in \Phi$ *as above.*

To ensure that the identity makes sense we first note that by (4.2) and the conditions on $Q$, we have $\ddot{Q}\varphi(x_i) \subset \hat{\mathfrak{A}}_i$. Now we quote [5, Lemma 3.6] and get (4.7) as an easy consequence.

For $i = 0, 1$ we define $\Phi_i$ to be the group of linear transformations of $\mathfrak{A}_i \times (\mathrm{Sym}(\mathfrak{A}_{1/2}, \sigma) \times \hat{\mathfrak{A}}_i)$ that are of the form $(x_i, Y, \hat{z}_i) \mapsto (\dot{Q}x_i, \dot{Q}_{1/2}Y\dot{Q}_{1/2}^\sigma, \ddot{Q}\hat{z}_i)$ where $Q = (\dot{Q}, \ddot{Q})$ is an element of $\Phi$ satisfying the conditions before (4.7). Finally, we put $M_i := (R_i, \varphi_i, \hat{R}_i)$.

We want to prove that $M_i$ is a morphism of homogeneous cones satisfying (M.5) such that in (M.4) we may choose $\Gamma = \Phi_i$.

For this we first derive a somewhat technical, but useful, result. We keep the notation introduced up to now. Especially, for a cone $C$ in $\mathfrak{A}_i$, the $\sigma_i$-dual cone for $C$ is simply denoted by $C^\sigma$.

LEMMA 4.1. *For* $i = 0, 1$, *we have*
(a) $[K_i]^\sigma = [K^\sigma]_i$, $\widetilde{[\gamma_i]} = [\tilde{\gamma}]_i$.
(b) $\widehat{G(M_i)} = [\widehat{G(M)}]_i$.

PROOF. Put $h := h_{\widehat{G(M)}}$ and $h_1 := h_{[\widehat{G(M)}]_i}$.
(a) From [5, Theorem 1.8], we get $K^\sigma = \bigcup_{x_{1/2} \in \mathfrak{A}_{1/2}} \exp A_{p_1}(x_{1/2})([K^\sigma]_1 + [K^\sigma]_0)$. We apply [5, Lemma 2.2], and derive

$$K = \bigcup_{x_{1/2} \in \mathfrak{A}_{1/2}} \exp A_{p_0}(x_{1/2})(h_1([K^\sigma]_1) + h_0([K^\sigma]_0)). \qquad (*)$$

We use the definition of $K_i$ and easily derive $h_i([K^\sigma]_i) = K_i$. But $h_i$ maps $[K^\sigma]_i$ onto $[K^\sigma]_i^\sigma$, so the first part of (a) is proved. To verify the second part we put $g_i := h_{G(M_i)}$ and recall $h_{G(M)} = h^{-1}$. The equality we want to prove is then equivalent to $\gamma(p_j + g_i^{-1}(x_i)) = \gamma(h(p_j + x_i))$, $i \neq j$, $x_i \in K_i^\sigma$. It is now easy to check that it suffices to prove $h_{G(M)}(x_1 + x_0) = h_{G(M_1)}(x_1) + h_{G(M_0)}(x_0)$ for all $x_1 + x_0 \in K_1 + K_0$. But this identity follows by a straightforward calculation, as in the proof of [5, Lemma 2.2(b)], starting with $\gamma(x) = \gamma(x_0 + x_1 - \frac{1}{2}p_1(x_{1/2} \cdot A_{1/2}(x_0)^{-1}))$, $x \in K$, an identity which itself is easily derived from $(*)$.

By the last lemma we may write $K_i^\sigma$ and $\tilde{\gamma}_i$ without causing any confusion. After these preparations we are going to prove

THEOREM 4.2. *Let* $M$ *be a morphism of homogeneous cones satisfying* (M.5). *Then for* $i = 0, 1$ *we have*
(a) $M_i$ *is a morphism of homogeneous cones that satisfies* (M.5).
(b) $\Phi_i$ *is a connected, trigonalizable Lie-subgroup of* $\Pi_{M_i}^\alpha$ *such that the projection of* $\Phi_i$ *onto* $\mathrm{Gl}\,\mathfrak{A}_i$ *acts transitively on* $K_i$.

(c) $\gamma_{G(M_i)} = g_i\gamma_i$ *for some* $g_i > 0$.

PROOF. To prove (a) and (b), we observe that, by the statements (4.4) to (4.7) and the definition of $\Phi_i$, it suffices to show that $\Phi_i$ is a connected, trigonalizable Lie-group such that the projection onto Gl $\mathfrak{A}_i$ acts transitively on $K_i$. For this, denote by $L_i$ the set of $Q \in \Phi$ that satisfy the conditions before (4.7); then it is easy to see that $\dot{L}_i|_{\mathfrak{A}_j}, j = 0, \frac{1}{2}, 1$, and $\ddot{L}_i$ are trigonalizable. It follows that $\Phi_i$ is trigonalizable. We put $D_i := \dot{L}_i|_{\mathfrak{A}_i}$ and see that $D_i$ coincides with the projection of $\Phi_i$ onto Gl $\mathfrak{A}_i$. Further, we apply [5, Theorem 6.1] to the triple $\widetilde{G(M)}$ of $\mathscr{F}$ and to $p = \tilde{c}$. It results that $D_i^{\sigma}$ operates transitively on $K_i^{\sigma}$. Therefore, $D_i$ acts transitively on $K_i$. It remains only to prove that $\Phi_i$ is connected. Since $\Phi_i$ is obviously a continuous image of $L_i$, it suffices to show that $L_i$ is connected. But, since $L_i$ operates transitively on $K_i$, the connected component $L_i^0$ of $L_i$ that contains Id, also acts transitively on $K_i$. Hence, if $L_i$ is not connected, we get a nontrivial $W \in L_i$ which satisfies $We = e$. But $L_i \subset \Phi$ and $\Phi$ is trigonalizable and connected, so Remark 2.2 gives a contradiction. For the demonstration of (c), by (1.1) and the definitions of $R_i$ and $\hat{R}_i$, we get as a first result that $\gamma_{G(M_i)}(x_i) = r_i(x)\hat{r}_i(\varphi_i(x_i)) = \iota(K_i; x_i)\hat{\eta}_i(\varphi(x_i))[\det A_{1/2}(x_i)]^{-1/2}$. On the other hand, we have $\gamma_i(x_i) = \gamma(p_{1-i} + x_i) = \iota(K; p_{1-i} + x_i)\hat{\eta}_i(\varphi(p_{1-i} + x_i)) = \iota(K; p_{1-i} + x_i)\hat{\eta}_i(\varphi(x_i))$. We consider the ratio $\psi(x_i) := \gamma_{G(M_i)}(x_i)[\gamma_i(x_i)]^{-1}$ and compute the value $\psi(Wp_i)$ for an arbitrary $W \in L_i$. It is easily seen that $\psi$ is constant and the proof is finished, since $D_i = L_i|_{\mathfrak{A}_i}$ operates transitively on $K_i$.

REMARK 4.3. By the lemma the triples $G(M_i)$ and $G(M)_i$ only differ by a constant factor at the functions. Hence the derived "objects", i.e., $h, H, \mathfrak{A}, \ldots$ are the same for both triples.

**5. $J$-morphisms.** As noted in the introduction, we specialize again. So we call a morphism $M = (F, \varphi, \hat{F})$ of homogeneous cones satisfying (M.5) a *$J$-morphism of homogeneous cones* iff

(M.6) *The algebra* $\hat{\mathfrak{A}}$ *is a Jordan-algebra.*

REMARK 5.1. (a) For a $J$-morphism $M = (F, \varphi, \hat{F})$ of homogeneous cones it can be shown, using [5, Theorems 1.6, 4.7], that $\hat{K}$ is the "positive cone" of $\hat{\mathfrak{A}}$ ($\hat{K} = Y_{\hat{\mathfrak{A}}}$ in the sense of [1, XI, §3]). Further, it is seen that $\hat{K}$ is the "sum of cones" $\hat{K}^{(i)}$ where the $\hat{K}^{(i)}$ are the positive cones of the simple ideals of $\hat{\mathfrak{A}}$. The function $\hat{\eta}$ splits into a product of functions $\hat{\eta}^{(i)}$ which are defined on $\hat{K}^{(i)}$. Finally, $\hat{\eta}^{(i)}$ is–up to a constant factor–a power of the invariant $\iota(\hat{K}^{(i)}; )$ of $\hat{K}^{(i)}$. From this we conclude that the algebras that are derived from $\hat{\eta}$ and from $\iota(\hat{K}; )$ are the same.

Hence it seems to be natural to put the condition that $\hat{\eta}$ equals–up to a constant factor–the invariant of $\hat{K}$.

But, in what follows we use, several times, the morphisms $M_i$ constructed in

§4. These $M_i$ are constructed in a canonical way, and $\hat{r}_i$ is not–up to a constant factor–the invariant of $K_{\hat{R}_i}$. So to simplify the arguments we just require (M.6).

(b) The definition of a $J$-morphism of homogeneous cones generalizes some notion of [11] and [12].

(c) As implicitly mentioned in (a), the algebra $\hat{\mathfrak{A}}$ is a formally-real Jordan-algebra, i.e. $x^2 + y^2 = 0$ implies $x = y = 0$. For results on formally-real Jordan-algebras we refer to [1, XI].

In the sequel the product in $\hat{\mathfrak{A}}$ is denoted by $(\hat{u}, \hat{v}) \mapsto \hat{u}\hat{v}$ and the product in $\mathfrak{A} = \mathfrak{A}_{G(M)}$ is denoted by $(u, v) \mapsto uv$. As the elements of $\hat{\mathfrak{A}}$ are marked by a "$\hat{\phantom{o}}$" or are of the form $\varphi(u)$, this will lead to no confusion. Finally, we use $x \cdot yz := x(yz)$.

Note that Remark 1.2 is still in effect.

LEMMA 5.2. *Let* $M = (F, \varphi, \hat{F})$ *be a* $J$*-morphism of homogeneous cones. Then for all* $W \in \mathrm{Aut}(K, \gamma)$, $T \in \mathrm{Lie\,Aut}(K, \gamma)$, $x \in K$ *and* $u, v, w \in V$ *we have*

(a) $\hat{\sigma}(\hat{h}(\varphi(Wx)), \varphi(Wv)) = \hat{\sigma}(\hat{h}(\varphi(x)), \varphi(v))$,

(b) $\hat{\sigma}(\hat{H}(\varphi(x))\varphi(Tx), \varphi(v)) = \hat{\sigma}(\hat{h}(\varphi(x)), \varphi(Tv))$,

(c) $\hat{\sigma}(\hat{H}(\varphi(x);\ \varphi(u))\varphi(Tx),\ \varphi(v)) + \hat{\sigma}(\hat{H}(\varphi(x))\varphi(Tu), \varphi(v)) + \hat{\sigma}(\hat{H}(\varphi(x))\varphi(u), \varphi(Tv)) = 0$,

(d) $\hat{\sigma}(\varphi(Te), \varphi(v)) = \hat{\sigma}(\hat{e}, \varphi(Tv))$,

(e) $2\hat{\sigma}(\varphi(u)\varphi(Te), \varphi(v)) = \hat{\sigma}(\varphi(Tu), \varphi(v)) + \hat{\sigma}(\varphi(u), \varphi(Tv))$,

(f) $\hat{\sigma}(\varphi(Tu), \varphi(v)\varphi(w)) + \hat{\sigma}(\varphi(Tw), \varphi(v)\varphi(u)) + \hat{\sigma}(\varphi(u)\varphi(w), \varphi(Tv)) = \hat{\sigma}(\varphi(u)\varphi(w), \varphi(v)\varphi(Te)) + \hat{\sigma}(\varphi(v)\varphi(u), \varphi(w)\varphi(Te)) + \hat{\sigma}(\varphi(v)\varphi(w), \varphi(u)\varphi(Te))$.

PROOF. (a) Because of $\eta_F = \iota(K_F;\ )$ we have $\mathrm{Aut}(K, \gamma) = \{W \in \mathrm{GL}\,V;\ WK = K,\ \hat{\eta}(\varphi(Wx)) = \alpha(W)\hat{\eta}(\varphi(x))$ for all $x \in K\}$. Now a differentiation of the identity $\log \hat{\eta}(\varphi(Wx)) = \log \alpha(W) + \log \hat{\eta}(\varphi(x))$ at $x$ in direction $v$ gives the assertion.

(b) Put $W := \exp tT$, insert in (a) and differentiate for $t$. Then put $t = 0$.

(c) Differentiate (b) at $x$ in direction $u$.

(d) and (e) follow from (b) and (c) with $x := e$.

(f) Differentiate (c) at $x = e$ in direction $w$ to see that $\hat{\sigma}(\frac{1}{2}\hat{H}(\hat{e};\ \varphi(u), \varphi(w))\varphi(Te), \varphi(v))$ is equal to the left-hand side of the equation (f). Now, as $\hat{\mathfrak{A}}$ is a Jordan-algebra, we conclude from [5, Theorem 4.7] that $\hat{\mathfrak{A}} = \hat{\mathfrak{S}} \subset \hat{\mathfrak{X}}$, so $\hat{A}(\hat{u}) \in \mathrm{Lie\,Aut}(\hat{K}, \hat{\eta})$ by the definition of $\hat{\mathfrak{X}}$. We use [7b, (1.18)] and the assertion follows.

In preparation for the next theorem we first prove

LEMMA 5.3. *Let* $M$ *be a* $J$*-morphism of homogeneous cones. Then the* $M_i$ *as constructed in* §4 *are also* $J$*-morphisms of homogeneous cones.*

PROOF. From Theorem 4.2 it is clear that we have only to check (M.6). But

from the definition of $\hat{r}_i$ before (4.6) we see that $\hat{\mathfrak{A}}$ is the product of algebras defined to the functions $A \mapsto (\det A)^{-1/2}$ and $\hat{x}_i \mapsto \hat{\eta}_i(\hat{x}_i)$ respectively.

The first function is a power of the invariant of $P(\mathfrak{A}_{1/2}, \sigma)$ and so produces a Jordan-algebra. As to the second function we may apply the results of [5, §2] (because of $\hat{\mathfrak{A}} = \hat{\mathfrak{S}}$) and see that $\mathfrak{A}_{\hat{r}_i}$ equals $\hat{\mathfrak{A}}_i$. So from the properties of a Peirce-decomposition we conclude that $\hat{\mathfrak{A}}_i$ is a Jordan-algebra [1, I, Satz 12.3].

As we go along, we often deal with $\mathfrak{X} = \mathfrak{X}_{G(M)}$. To ensure that the next theorem makes sense, we note that $\mathfrak{X}_R$, $R$ a triple of $\mathfrak{F}$, is a Jordan-algebra where the algebra structure is inherited from $\mathfrak{A}_R$ (see [5, Corollary 5.4]).

**THEOREM 5.4.** *Let $M = (F, \varphi, \hat{F})$ be a J-morphism of homogeneous cones. Then*

(a) $\varphi: \mathfrak{X}_{G(M)} \to \hat{\mathfrak{A}}$ *is a homomorphism of Jordan-algebras.*

(b) $\hat{\sigma}(\hat{e}, \varphi(x^2 \cdot v)) = \hat{\sigma}(\hat{e}, \varphi(x \cdot xv)) = \hat{\sigma}(\varphi(x)^2, \varphi(v))$ *for all $x \in \mathfrak{X}_{G(M)}$, $v \in \mathfrak{A}$.*

PROOF. We prove the assertion in two steps.

(1) Assume additionally that the mapping $x \mapsto \hat{\sigma}(\hat{e}, \varphi(x))$ is an associative linear form of the Jordan-algebra $\mathfrak{X} = \mathfrak{X}_{G(M)}$: In part (e) of Lemma 5.2 we put $u := x$ and $T = A(x)$ and get for all $v \in \mathfrak{A}$, $x \in \mathfrak{X}$

(a) $2\hat{\sigma}(\varphi(x)^2, \varphi(v)) = \hat{\sigma}(\varphi(x^2), \varphi(v)) + \hat{\sigma}(\varphi(x), \varphi(xv))$.

As $x \in \mathfrak{X}$ we have $x^2 \in \mathfrak{X}$ by [5, Corollary 5.4]. So in part (d) of Lemma 5.2 we put $T = A(x^2)$ and obtain

(b) $\hat{\sigma}(\varphi(x^2), \varphi(v)) = \hat{\sigma}(\hat{e}, \varphi(x^2v))$.

Analogously we replace $T$ by $A(x)$ and $v$ by $xv$ and get

(c) $\hat{\sigma}(\varphi(x), \varphi(xv)) = \hat{\sigma}(\hat{e}, \varphi(x \cdot xv))$.

Putting (a), (b) and (c) together it follows that

(d) $2\hat{\sigma}(\varphi(x)^2, \varphi(v)) = \hat{\sigma}(\hat{e}, \varphi(x^2 \cdot v + x \cdot xv))$, $x \in \mathfrak{X}$, $v \in \mathfrak{A}$.

Now let $v \in \mathfrak{X}$, then using the additional assumption above we have

(e) $\hat{\sigma}(\varphi(x)^2, \varphi(v)) = \hat{\sigma}(\hat{e}, \varphi(x^2 \cdot v)) = \hat{\sigma}(\varphi(x^2), \varphi(v))$, $x, v \in \mathfrak{X}$.

Denote by $\hat{U}$ the orthogonal complement of $\varphi(\mathfrak{X})$ in $\hat{\mathfrak{A}}$ with respect to $\hat{\sigma}$. Then from (e) we conclude

(f) $\varphi(x)^2 = \varphi(x^2) + \hat{u}$, $\hat{u} \in \hat{U}$.

We now apply part (f) of Lemma 5.2. We put $u = v = w = x$ and $T = A(x)$. We obtain

(g) $\hat{\sigma}(\varphi(x^2), \varphi(x)^2) = \hat{\sigma}(\varphi(x)^2, \varphi(x)^2)$, $x \in \mathfrak{X}$.

So as $x^2 \in \mathfrak{X}$ we have by (f)

(h) $\hat{\sigma}(\varphi(x^2), \varphi(x^2)) = \hat{\sigma}(\varphi(x)^2, \varphi(x)^2)$, $x \in \mathfrak{X}$.

From this and (f) we conclude (a). Part (b) follows from (d) with (a).

(2) The general case is proved by induction on $n = \dim \mathfrak{A}$. To be clear we index the occurring algebras by the corresponding triples of $\mathfrak{F}$. To be brief, we write $G$ instead of $G(M)$.

For $n = 1$ there is nothing to prove. So assume $n > 1$. We distinguish two cases.

1. *Case*: $\mathfrak{A}_G = \mathfrak{S}_G$. Here we first show that $\mathfrak{S}_G$ equals $\mathfrak{S}_F$ as an algebra. To get this denote the left multiplications of $\mathfrak{A}_F$ by $B(x)$ and the left multiplications of $\mathfrak{A}_G$ by $A(x)$ and observe that for all $d \in \mathfrak{S}_G$, $x \in V$, we have $A_d(x) \in \mathrm{Lie}\ \mathrm{Aut}(K, \gamma)$ [5, §1]. Now as $\eta_F$ is the invariant of $K_F = K_G = K$ we conclude $\mathrm{Aut}(K, \gamma) \subset \mathrm{Aut}(K, \eta_F)$. Therefore, for all $d \in \mathfrak{S}_G$ all left multiplications of the commutative algebra $(\mathfrak{A}_G)_d$ are elements of $\mathrm{Lie}\ \mathrm{Aut}(K, \eta_F)$. We apply [10, Lemma 1.1 and Satz 5.10] and obtain a $f \in \mathfrak{S}_F$ such that $A_d(x) = B_f(x)$ for all $x \in V$. As $e = e_F = e_G$ is the unit for both algebras, we derive $d = A_d(e)e = B_f(e)e = f$. Therefore $V = \mathfrak{A}_G = \mathfrak{S}_G \subset \mathfrak{S}_F \subset V$. Further, $A(d) = A_d(e) = B_d(e) = B(d)$ proves $\mathfrak{A}_F = \mathfrak{A}_G$ as algebras. From (1.2) we get $\sigma_G(e, u) = \sigma_F(e, u) + \hat{\sigma}(\hat{e}, \varphi(u))$ for all $u \in V$. Since $\mathfrak{X}_G = \mathfrak{A}_G = \mathfrak{A}_F$ we see that $u \mapsto \hat{\sigma}(\hat{e}, \varphi(u))$ is an associative linear form on $\mathfrak{X}_G$ (see [5, (1.11)]), and the assertion follows by (1).

2. *Case*: $\mathfrak{A}_G \neq \mathfrak{S}_G$. In this case we also have $\mathfrak{A}_{\tilde{G}} \neq \mathfrak{S}_{\tilde{G}}$ by [5, Theorem 4.7]. Using [7b, §5] (see also [5, Theorem 1.1]) we get $0 < \dim \mathfrak{S}_{\tilde{G}} < \dim \mathfrak{A}_{\tilde{G}}$. We consider the morphisms of homogeneous cones $M_i$ as constructed in §4. From Lemma 5.3 we conclude by the induction hypothesis that $\varphi_i: \mathfrak{X}_{G(M_i)} \to \mathrm{Sym}(\mathfrak{A}_{1/2}, \sigma) \times \hat{\mathfrak{A}}_i$ is a homomorphism of Jordan-algebras, and obviously so is $\varphi: \mathfrak{X}_{G(M_i)} \to \hat{\mathfrak{A}}_i$. Here, by Remark 4.3, the equations $\mathfrak{X}_{G(M_i)} = \mathfrak{X}_{G(M)_1} = \mathfrak{S}_{G(M)_1} = \mathfrak{A}_1$ and $\mathfrak{X}_{G(M_0)} = \mathfrak{X}_{G(M)_0}$ hold. Further, from [5, Theorem 3.3(f)], it is clear that $\mathfrak{X}_{G(M)} \subset \mathfrak{A}_1 + \mathfrak{A}_0$. So we obtain $\mathfrak{X}_{G(M)} = \mathfrak{X}_{\widetilde{(GM)}} = \mathfrak{A}_1 + (\mathfrak{X}_{G(M)} \cap \mathfrak{A}_0)$. From [5, Corollary 6.5], we conclude $\mathfrak{X}_{G(M)} \cap \mathfrak{A}_0 \subset \mathfrak{X}_{G(M)_0}$. Hence $\varphi: \mathfrak{X}_{G(M)} \to \hat{\mathfrak{A}}$ is a homomorphism of algebras. Consequently $x \mapsto \hat{\sigma}(\hat{e}, \varphi(x))$ defines an associative linear form of $\mathfrak{X}_{G(M)}$. The assertion now follows by (1).

Another important property of $\varphi$ is stated in the next theorem.

**THEOREM 5.5.** *Let $M = (F, \varphi, \hat{F})$ be a J-morphism of homogeneous cones. Then $\varphi(xv) = \varphi(x)\varphi(v)$ for all $x \in \mathfrak{X}_{G(M)}$, $v \in V$.*

PROOF. The proof proceeds by induction on $n = \dim \mathfrak{A}$. For $\dim \mathfrak{A} = 1$ we have nothing to show. So assume $n > 1$. Here we consider two cases.

1. *Case*: $\mathfrak{A} = \mathfrak{S}$. Here the assertion follows by Theorem 5.4.

2. *Case*: $\mathfrak{A} \neq \mathfrak{S}$. We use the J-morphisms $M_i$ of homogeneous cones as constructed in §4. Then by induction we see that $\varphi(x_i v_i) = \varphi(x_i)\varphi(v_i)$ for all $x_i \in \mathfrak{X}_{G(M)_i}$, $v_i \in \mathfrak{A}_i$ (for more details compare the proof of Theorem 5.4). Because of $\mathfrak{X}_{G(M)} = \mathfrak{A}_1 + (\mathfrak{X}_{G(M)} \cap \mathfrak{A}_0)$ and $\mathfrak{X}_{G(M)} \cap \mathfrak{A}_0 \subset \mathfrak{X}_{G(M)_0}$ we have only to prove

$$\varphi(x_i v_{1/2}) = \varphi(x_i)\varphi(v_{1/2}) \quad \text{for all } x_i \in \mathfrak{X}_{G(M)} \cap \mathfrak{A}_i, v_{1/2} \in \mathfrak{A}_{1/2}. \quad (*)$$

For this we first note that

$$x_i^2 \cdot v_{1/2} = 2x_i \cdot x_i v_{1/2} \quad \text{and} \quad \varphi(x_i)^2 \cdot \varphi(v_{1/2}) = 2\varphi(x_i) \cdot \varphi(x_i)\varphi(v_{1/2}). \quad (1)$$

Here the first equation is obtained by putting $x := x_i$ and $v := v_{1/2}$ in part b) of [**7b**, Satz 3.1], and applying the resulting equation to $p_j$, $j \neq i$. The second equation follows analogously (and is well known from Jordan-theory). Now in Lemma 5.2(f) we put $v = w = v_{1/2}$, $u = x_i$, $T = A(x_i)$ and get

$$\hat{\sigma}\big(\varphi(v_{1/2})\varphi(x_i), \varphi(x_i v_{1/2})\big) = \hat{\sigma}\big(\varphi(x_i)\varphi(v_{1/2}), \varphi(x_i)\varphi(v_{1/2})\big). \tag{2}$$

Now insert $u = v = v_{1/2}$ and $T = A(x_i^2)$ in Lemma 5.2(e), and use (1) and Theorem 5.4(a). It follows

$$\hat{\sigma}\big(\varphi(v_{1/2})\varphi(x_i), \varphi(v_{1/2})\varphi(x_i)\big) = \hat{\sigma}\big(\varphi(x_i \cdot x_i v_{1/2}), \varphi(v_{1/2})\big). \tag{3}$$

But putting $v = x_i v_{1/2}$, $u = v_{1/2}$, $T = A(x_i)$ in Lemma 5.2(e), gives

$$2\hat{\sigma}\big(\varphi(v_{1/2})\varphi(x_i), \varphi(x_i v_{1/2})\big)$$
$$= \hat{\sigma}\big(\varphi(x_i v_{1/2}), \varphi(x_i v_{1/2})\big) + \hat{\sigma}\big(\varphi(v_{1/2}), \varphi(x_i \cdot x_i v_{1/2})\big). \tag{4}$$

We use (2) and conclude that the left-hand side of (4) is equal to $2\hat{\sigma}(\varphi(v_{1/2})\varphi(x_i), \varphi(v_{1/2})\varphi(x_i))$. With (3) we thus get out of (4)

$$\hat{\sigma}\big(\varphi(v_{1/2})\varphi(x_i), \varphi(v_{1/2})\varphi(x_i)\big) + \hat{\sigma}\big(\varphi(v_{1/2}), \varphi(x_i \cdot x_i v_{1/2})\big)$$
$$= \hat{\sigma}\big(\varphi(x_i v_{1/2}), \varphi(x_i v_{1/2})\big) + \hat{\sigma}\big(\varphi(v_{1/2}), \varphi(x_i \cdot x_i v_{1/2})\big).$$

It follows

$$\hat{\sigma}\big(\varphi(v_{1/2})\varphi(x_i), \varphi(v_{1/2})\varphi(x_i)\big) = \hat{\sigma}\big(\varphi(x_i v_{1/2}), \varphi(x_i v_{1/2})\big). \tag{5}$$

Now put $\hat{a} := \varphi(v_{1/2})\varphi(x_i)$ and $\hat{b} := \varphi(x_i v_{1/2})$; so from (2) and (5) we have

$$\hat{\sigma}(\hat{a}, \hat{b}) = \hat{\sigma}(\hat{a}, \hat{a}) = \hat{\sigma}(\hat{b}, \hat{b}). \tag{6}$$

By the Cauchy-Schwarz inequality we conclude $\hat{a} = \hat{b}$ and the theorem is proved.

As a consequence we derive some properties of $A_{1/2}$ and $\mathfrak{X}$.

THEOREM 5.6. *Let $M$ be a $J$-morphism of homogeneous cones, and let $\mathfrak{A} = \mathfrak{A}_1 + \mathfrak{A}_{1/2} + \mathfrak{A}_0$ be the Peirce-decomposition of $\mathfrak{A}_{\widetilde{G(M)}}$ with respect to the unit $\tilde{c}$ of $\mathfrak{S}_{\widetilde{G(M)}}$. Then*

(a) $A_{1/2}(x_0 v_0) = A_{1/2}(x_0)A_{1/2}(v_0) + A_{1/2}(v_0)A_{1/2}(x_0)$ *for all* $x_0 \in \mathfrak{X}_{G(M)_0}$, $v_0 \in \mathfrak{A}_0$.

(b) $\mathfrak{X}_{G(M)} = \mathfrak{A}_1 + \mathfrak{X}_{G(M)_0}$.

PROOF. (a) Consider $M_0$ and conclude $\varphi_0(x_0 v_0) = \varphi_0(x_0)\varphi_0(v_0)$, $x_0 \in \mathfrak{X}_{G(M)_0}$, $v_0 \in \mathfrak{A}_0$ by Theorem 5.5. Compare now the first components of both sides and note that the algebra on $\mathrm{Sym}(\mathfrak{A}_{1/2}, \sigma)$ derived for $A \mapsto (\det A)^{-1/2}$ at the point $\frac{1}{2}$ Id is $(X, Y) \mapsto XY + YX$.

(b) A straightforward calculation shows that $A(x_0)$ is a derivation of the algebra $\mathfrak{A}_{\tilde{c}}$ iff $x_0(u_{1/2} v_{1/2}) = p_0(x_0 u_{1/2} \cdot v_{1/2}) + p_0(u_{1/2} \cdot x_0 v_{1/2})$ for all $u_{1/2}, v_{1/2} \in \mathfrak{A}_{1/2}$. This identity is easily verified for $x_0 \in \mathfrak{X}_{G(M)_0}$ from (a) and

the associativity of $\sigma$, [5, (1.11)]. Hence $W_t := \exp tA(x_0)$, $t \in \mathbf{R}$, is an automorphism of the algebra $\mathfrak{A}_{\tilde{c}}$ for all $x_0 \in \mathfrak{X}_{G(M)_0}$. Since $\mathfrak{X}_{G(M)_0} = \mathfrak{X}_{G(M_0)} = \mathfrak{X}_{\widetilde{G(M_0)}} = \mathfrak{X}_{[\widetilde{G(M)}]_0}$, by Lemma 4.1, we have $W_t|_{\mathfrak{A}_0} \in \mathrm{Aut}([K^\sigma]_0, [\tilde{\gamma}]_0)$. We use these two properties, apply [5, Theorem 1.8], to $K^\sigma$, $p = \tilde{c}$ and get $W_t \in \mathrm{Aut}(K^\sigma, \tilde{\gamma})$. This implies $x_0 \in \mathfrak{X}_{\widetilde{G(M)}} = \mathfrak{X}_{G(M)}$ and so $\mathfrak{A}_1 + \mathfrak{X}_{G(M_0)} \subset \mathfrak{X}_{G(M)}$ follows. The other inclusion always being satisfied (compare, e.g., the proof of Theorem 5.4), the theorem is proved.

**6. The "$\mathfrak{R}$-decomposition" $\mathcal{Q}_M$.** For a $J$-morphism $M = (F, \varphi, \hat{F})$ of homogeneous cones we define inductively

$$N^{(1)} := M, \quad N^{(i+1)} := (N^{(i)})_0, \quad i > 1. \tag{6.1}$$

Further, we put

$$N^{(i)} := (F^{(i)}, \varphi^{(i)}, \hat{F}^{(i)}), \quad i > 1, \tag{6.2}$$

and

$$F^{(i)} := \langle K^{(i)}, \eta^{(i)}, e^{(i)} \rangle, \quad \hat{F}^{(i)} := \langle \hat{K}^{(i)}, \hat{\eta}^{(i)}, \hat{e}^{(i)} \rangle, \quad i > 1. \tag{6.3}$$

$$G^{(i)} := G(N^{(i)}), \quad i > 1. \tag{6.4}$$

Finally, we denote the unit of $\mathfrak{S}_{G^{(i)}}$ by $e_{ii}$ and the last integer $i$ such that $V_{F^{(i)}} \neq 0$ by $q_M$. We usually write $q$ instead of $q_M$.

LEMMA 6.1. *Let $M$ be a $J$-morphism of homogeneous cones. Then*
(a) $\mathfrak{X}_{G^{(1)}} = \mathfrak{S}_{G^{(1)}} + \mathfrak{X}_{G^{(2)}} = \bigoplus_{i=1}^{q} \mathfrak{S}_{G^{(i)}}$.
(b) $e_{11}, \ldots, e_{qq} \in \mathfrak{X}_{G^{(1)}}$ *is a CSI.*

PROOF. (a) follows immediately from Lemma 5.3 and Theorem 5.6(b). Part (b) is a consequence of (a).

Using the last lemma, we form the Peirce-decomposition

$$\mathfrak{A} = \bigoplus_{1 \leq i \leq j \leq q} \mathfrak{A}_{ij}$$

of $\mathfrak{A} = \mathfrak{A}_{G(M)} = \mathfrak{A}_{G^{(1)}}$ with respect to $e_{11}, \ldots, e_{qq}$ and put

$$\mathcal{Q}_M := (\mathfrak{A}_{ij}; 1 \leq i, j \leq q). \tag{6.5}$$

In the sequel we use the notion of a "$q$-$\mathfrak{R}$-decomposition". It has been introduced in [4]. For the convenience of the reader we recall its definition for algebras of type $\mathfrak{A}_R$, $R$ a triple of $\mathfrak{F}$. Note that the definitions also make sense in the case of algebras which have a $q$-Peirce-decomposition [4, §1].

We abbreviate $\mathfrak{E} := \mathfrak{A}_R$ and denote the left multiplications of $\mathfrak{E}$ by $C(x)$, $x \in \mathfrak{E}$. We start with a $q$-Peirce-decomposition $\mathcal{C} = (\mathfrak{E}_{ij}; 1 \leq i, j \leq q)$ of $\mathfrak{E}$ with respect to some CSI $c_1, \ldots, c_q$.

For $1 \leq k \leq q$ we form the subalgebras $\mathfrak{E}^{(k)} := \bigoplus_{k \leq r \leq s \leq q} \mathfrak{E}_{rs}$ and denote the left multiplications of the mutant $\mathfrak{E}_{c_r}^{(k)}$ of $\mathfrak{E}^{(k)}$ by $c_r$, $k \leq r \leq q$, by $C_{rr}^{(k)}(x)$, $x \in \mathfrak{E}^{(k)}$. For $k = 1$ we omit the superscript "(1)". We say that $\mathcal{C}$ is a $q$-$\mathfrak{R}$-*decomposition* iff

($\Re$.1) $\mathfrak{C}_{C_k}^{(k)}$ is a Jordan-algebra for all $1 \leqslant k \leqslant q$,

($\Re$.2) $C_{rr}^{(k)}(x_{rs})$ is a derivation of $\mathfrak{C}_{C_k}^{(k)}$ for all $1 \leqslant k < r \leqslant s \leqslant q$ and all $x_{rs} \in \mathfrak{C}_{rs}$.

Sometimes we just say $\Re$-decomposition instead of $q$-$\Re$-decomposition. We note that in [4] we have constructed a *homogeneous cone* to each $\Re$-decomposition, i.e. a regular cone $Y$ for which $\langle Y, \iota(Y; \,), e \rangle$ is a triple of $\mathfrak{F}$ for some $e \in Y$. We will use the details of this construction in §7 where we come back to this situation.

Our main goal in this section is to prove that $\mathfrak{C}_M$ is a $q$-$\Re$-decomposition. This result is contained in Theorem 6.3.

To prepare this theorem we first prove

LEMMA 6.2. *Let* $M = (F, \varphi, \hat{F})$ *be a* $J$-*morphism of homogeneous cones and* $\Phi$ *a connected, trigonalizable subgroup of* $\Pi_M^a$ *such that* $\dot{\Phi}$ *operates transitively on* $K$. *Then*

(a) $A(e_{11}), \ldots, A(e_{qq}) \in \mathrm{Lie}\ \dot{\Phi}$.

(b) $A_{nn}(x_{sn}) \in \mathrm{Lie}\ \dot{\Phi}$ *for all* $1 \leqslant s < n \leqslant q$, $x_{sn} \in \mathfrak{A}_{sn}$.

PROOF. We prove the assertion by induction on $q$. For $q = 1$ we are done, by (4.1). For $q > 1$ we use Theorem 4.2(c) and see, by the induction hypothesis, that $A(e_{jj})|_{\mathfrak{A}_0}$, $j = 2, \ldots, q$, and $A_{nn}(x_{sn})|_{\mathfrak{A}_0}$, $2 \leqslant s < n \leqslant q$, $x_{sn} \in \mathfrak{A}_{sn}$, are elements of the projection of $\mathrm{Lie}\ \dot{\Phi}_0$ onto $\mathrm{End}\ \mathfrak{A}_0$. Now we apply [5, Theorem 6.1] to $\widetilde{G(M)}$ and $p = \tilde{c}$ and get that there exists a unique element $T \in \mathrm{Lie}\ \dot{\Phi}$ with $T\mathfrak{A}_j \subset \mathfrak{A}_j$, $j = 0, \frac{1}{2}, 1$, and $T\mathfrak{A}_1 = 0$ such that $T|_{\mathfrak{A}_0} = A(e_{jj})|_{\mathfrak{A}_0}$ resp. $T|_{\mathfrak{A}_0} = A_{nn}(x_{sn})|_{\mathfrak{A}_0}$. Choose $S \in \mathrm{Lie}\ \dot{\Phi}_0$ such that $A(e_{jj})|_{\mathfrak{A}_0}$ resp. $A_{nn}(x_{sn})|_{\mathfrak{A}_0}$ is the first component of $S$. Then Theorem 3.1 shows that the "second" component of $S$ is equal to $(J(A_{1/2}(e_{jj})), \hat{A}(\varphi(e_{jj})))$ resp. $(J_{A_{1/2}(e_{nn})}(A_{1/2}(x_{sn})), \hat{A}_{\varphi(e_{nn})}(\varphi(x_{sn})))$ where $J(R)$ is the left multiplication in the algebra on $\mathrm{Sym}(\mathfrak{A}_{1/2}, \sigma)$ defined for the function $A \mapsto (\det A)^{-1/2}$ at the point $\frac{1}{2}\mathrm{Id}$. As noted in the proof of Theorem 5.6, this product is $(X, Y) \mapsto XY + YX$. Now a comparison with (4.7) gives in the first case $T|_{\mathfrak{A}_{1/2}} = A_{1/2}(e_{jj})$ and because of $T|_{\mathfrak{A}_1} = 0 = A(e_{jj})|_{\mathfrak{A}_1}$ we see $T = A(e_{jj})$, $j = 2, \ldots, q$. The case $j = 1$ is clear from (4.1). On the other hand, a straightforward computation gives $J_X(Y)Z = 2YXZ + 2ZXY$; hence by a comparison with (4.7) it follows $2A_{1/2}(x_{sn})A_{1/2}(e_{nn}) = T|_{\mathfrak{A}_{1/2}}$. Now we use Lemma 6.1(b) and Theorem 5.6(a), to calculate directly $A_{nn}(x_{sn})|_{\mathfrak{A}_{1/2}} = 2A_{1/2}(x_{sn})A_{1/2}(e_{nn})$. As $T|_{\mathfrak{A}_1} = 0 = A_{nn}(x_{sn})|_{\mathfrak{A}_1}$ we get $T = A_{nn}(x_{sn})$. In the case $s = 1$, we observe that $A_{11}(x_{1n}) \in \mathrm{Lie}\ \dot{\Phi}^\sigma$ by [5, Theorem 6.1] applied to $\widetilde{G(M)}$ and $p = \tilde{c}$. So $[A_{11}(x_{1n})]^\sigma = A_{nn}(x_{1n}) \in \widetilde{\mathrm{Lie}\ \dot{\Phi}}$ by [4, Folgerung 1.2]. The lemma is proved.

THEOREM 6.3. *Let* $M$ *be a* $J$-*morphism of homogeneous cones. Then*

(a) $\mathfrak{C}_M$ *is a* $q$-$\Re$-*decomposition of* $\mathfrak{A}_{G^{(1)}}$.

(b) $\mathfrak{S}_{G^{(i)}} = \mathfrak{A}_{ii}$ *for* $1 \leqslant i \leqslant q$.

PROOF. First we note that (b) is clear from the construction of $N^{(i+1)}$ out of $N^{(i)}$. Further, because of $e_{ii}$ being the unit of $\mathfrak{S}_{G^{(i)}}$ we see (from [10] or [5, Theorem 1.2]) that the "mutation" $[\mathfrak{A}_{\widehat{G^{(i)}}}]_{e_{ii}}$ of $\mathfrak{A}_{\widehat{G^{(i)}}}$ is a Jordan-algebra. The proof of (a) is now achieved by induction on $q$. For $q = 1$ the assertion is clear. So let $q > 1$. From the definition above we derive that only ($\mathfrak{R}$.2) is left to verify. Using the induction hypothesis, we have just to prove

(∗) $A_{ss}(x_{sn})$ is a derivation of $\mathfrak{A}_{e_{11}}$ for $2 \leqslant s \leqslant n \leqslant q$ and all $x_{sn} \in \mathfrak{A}_{sn}$.

For $s = n$ we use (b), Theorem 5.6(a), and [4, Lemma 2.4]. For $s < n$ we conclude from Lemma 6.2 that $A_{ss}(x_{sn}) \in \text{Lie } \dot{\Phi}^{\sigma} \subset \text{Lie Aut}(K^{\sigma}, \tilde{\gamma})$. Further, $[A_{ss}(x_{sn})]^{\sigma}e_{11} = A_{nn}(x_{sn})e_{11} = A_{nn}(e_{11})x_{sn} = 0$ (see, e.g., [4, Lemma 1.1]). Hence from [5, Lemma 1.4] the assertion follows.

**7. Description of** Lie Aut$(K, \gamma)$. Let $R$ be a triple of $\mathfrak{F}$ and $a \in K_R$; then we denote the isotropy subgroup of Aut$(K_R, \eta_R)$ at the point $a$ by Aut$_a(K_R, \eta_R)$. As noted in the introduction we describe in this section Lie Aut$_e(K, \gamma)$ and Lie Aut$(K, \gamma)$ (for a triple of type $G(M)$). As mentioned in §4 we have Lie Aut$(K, \gamma)^{\sigma} = $ Lie Aut$(K^{\sigma}, \tilde{\gamma})$, hence it suffices to describe Lie Aut$(K^{\sigma}, \tilde{\gamma})$.

LEMMA 7.1. *Let* $M = (F, \varphi, \hat{F})$ *be a J-morphism of homogeneous cones. Then for* $W \in \text{GL } V$ *these are equivalent*:

(1) $W \in \text{Aut}_e(K, \gamma)$;

(2) $W$ *is an automorphism of the algebra* $\mathfrak{A}$ *and* $We_{ii} = e_{ii}$ *for* $1 \leqslant i \leqslant q$.

PROOF. (1) $\Rightarrow$ (2). From [5, (1.10)], we get $WW^{\sigma} = \text{Id}$; further, we obtain

$$WA(u)W^{-1} = -\frac{1}{2}\frac{d}{dt}H(W(e + tu))|_{t=0} = A(Wu).$$

Hence $W$ is an automorphism of the algebra $\mathfrak{A}$. To prove the second statement we first observe that for $x \in \tilde{\mathfrak{S}} = \mathfrak{S}_{\widehat{G(M)}}$ we have $A_{W_x}(Wv) = WA_x(v)W^{-1} \in \text{Lie Aut}(K^{\sigma}, \tilde{\gamma})$ for all $v \in V$. By definition of $\tilde{\mathfrak{S}}$ we get $W\tilde{\mathfrak{S}} \subset \tilde{\mathfrak{S}}$. So $W|_{\tilde{\mathfrak{S}}}$ is an automorphism of the Jordan-algebra $\tilde{\mathfrak{S}}$ and it results $We_{11} = e_{11}$. From this we conclude $W\mathfrak{A}_j \subset \mathfrak{A}_j$, for $j = 0, \frac{1}{2}, 1, \mathfrak{A}_j = \mathfrak{A}_j(\tilde{c})$. Further, we use Remark 4.3 to get $W \in \text{Aut}_{p_0}(K_0, \gamma_{M_0}), p_0 := e - \tilde{c}$. Now the assertion follows easily by induction on $q$.

(2) $\Rightarrow$ (1). Again we use induction on $q$. The case $q = 1$ is known from Jordan-theory [1, XI, Satz 4.5]. For $q > 1$ we conclude from $W\tilde{c} = We_{11} = e_{11}$ that $W\mathfrak{A}_j \subset \mathfrak{A}_j, \mathfrak{A}_j = \mathfrak{A}_j(\tilde{c})$. By Remark 4.3, $W$ is an automorphism of the algebra $\mathfrak{A}_i = \mathfrak{A}_{G(M_i)}$, $i = 0, 1$, and leaves fixed the $e_{jj}$. So by induction hypothesis we see $W|_{\mathfrak{A}_i} \in \text{Aut}_{p_i}(K_i, \gamma_i)$, $p_1 := \tilde{c}$, $p_0 := e - \tilde{c}$. Since $\text{Aut}_{p_i}(K_i, \cdot \gamma_i) = \text{Aut}_{p_i}(K_i^{\sigma}, \tilde{\gamma}_i)$ we conclude from [5, Theorem 1.8], applied to $\widetilde{G(M)}$ and $p = \tilde{c}$ that $W \in \text{Aut}_e(K^{\sigma}, \tilde{\gamma}) = \text{Aut}_e(K, \gamma)$.

We now aim to give a detailed description of Lie Aut$_e(K, \gamma)$. To do so, we

use the vector spaces $\mathfrak{g}_{ij}$ of endomorphisms associated to the $q$-$\mathfrak{K}$-decomposition $\mathcal{C}_M$.

We remind the reader of the definition of the $\mathfrak{g}_{ij} = \mathfrak{g}_{ij}^{\mathcal{C}}$, $1 \leqslant i \leqslant j \leqslant q$, where $\mathcal{C}$ is any $q$-$\mathfrak{K}$-decomposition of some algebra $\mathfrak{C}$ (we use the notations of §6).

For $i = j$ we define $\mathfrak{g}_{ii}$ to be the Lie-algebra which is generated by the set $\{C(x_{ii}); x_{ii} \in \mathfrak{C}_{ii}\}$ of endomorphisms of the vectorspace $\mathfrak{C}$. For $i < j$ we put $\mathfrak{g}_{ij} := \{C_{ii}(x_{ij}); x_{ij} \in \mathfrak{C}_{ij}\}$.

It has been proved in [4, Satz 3.3] that the sum $\mathfrak{g} = \mathfrak{g}^{\mathcal{C}}$ of the vector-spaces $\mathfrak{g}_{ij}$ is direct and forms a Lie-subalgebra of $\mathrm{End}_R \mathfrak{C}$. Further, there have been computed the commutators of elements of $\mathfrak{g}$.

We mention that for a $q$-$\mathfrak{K}$-decomposition $\mathcal{C}$ one can form the "*dual*" $q$-$\mathfrak{K}$-decomposition $\mathcal{C}^*$. If $\mathcal{C}$ is defined via the CSI $e_{11}, \ldots, e_{qq}$ then $\mathcal{C}^*$ is defined via the CSI $e_{qq}, \ldots, e_{11}$. The spaces $\mathfrak{g}_{ij}^{\mathcal{C}^*}$ are abbreviated by $\mathfrak{g}_{ij}^*$ if no confusion can arise. For more details see [4, §6].

To get a description of Lie $\mathrm{Aut}_e(K, \gamma)$ we define with respect to $\mathcal{C}_M$,

$$\mathfrak{a}_{ii} := \{T \in \mathfrak{g}_{ii}; Te_{ii} = 0\}, \qquad 1 \leqslant i \leqslant q. \tag{7.1}$$

$$\mathfrak{a}_R := \{T \in \mathrm{End}\, V; T \text{ is a derivation of } \mathfrak{A}_{G(M)}, T\mathfrak{X}_{G(M)} = 0\}. \tag{7.2}$$

It is clear that $\mathfrak{a}_{ii}$ and $\mathfrak{a}_R$ are Lie-algebras.

THEOREM 7.2. *Let $M = (F, \varphi, \hat{F})$ be a J-morphism of homogeneous cones. Then* Lie $\mathrm{Aut}_e(K, \gamma) = \bigoplus_{i=1}^{q} \mathfrak{a}_{ii} \oplus \mathfrak{a}_R$ *(direct sum of Lie-algebras).*

PROOF. (a) Let $T_{ii} \in \mathfrak{a}_{ii}$, $T_R \in \mathfrak{a}_R$ such that $\sum_{i=1}^{q} T_{ii} + T_R = 0$. Then we have $T_{ii}\mathfrak{A}_{jj} = 0$ for all $1 \leqslant i, j \leqslant q$. From [4, Lemmata 3.1, 2.3], we conclude $T_{ii} = 0$. Hence $T_R = 0$ and the sum is direct as the sum of vector spaces. By [4, Folgerung 1.2], we have $[\mathfrak{a}_{ii}, \mathfrak{a}_{jj}] = 0$ for $i \neq j$. Finally, we note that $[A(x_{ii}), T_R] = 0$ for all $x_{ii} \in \mathfrak{A}_{ii}$, $T_R \in \mathfrak{A}_R$, to get $0 = [\mathfrak{g}_{ii}, \mathfrak{a}_R] \supset [\mathfrak{a}_{ii}, \mathfrak{a}_R]$.

(b) The inclusion "$\supset$" is clear for the $\mathfrak{a}_{ii}$ and follows from Lemma 7.1 for $\mathfrak{a}_R$. Let now $T \in$ Lie $\mathrm{Aut}_e(K, \gamma) =$ Lie $\mathrm{Aut}_e(K^o, \tilde{\gamma})$. From Lemma 7.1 we get $Te_{ii} = 0$ for $1 \leqslant i \leqslant q$. So by the definition of the Peirce-spaces $\mathfrak{A}_{ij}$ it follows that $T\mathfrak{A}_{ij} \subset \mathfrak{A}_{ij}$ for $1 \leqslant i, j \leqslant q$. Since $T|_{\mathfrak{A}_{ii}}$ is a derivation of the formally-real Jordan-algebra $\mathfrak{A}_{ii}$, we find by [1, IX, §3], $x_{ii}^{(k)}, y_{ii}^{(k)} \in \mathfrak{A}_{ii}$ such that $T|_{\mathfrak{A}_{ii}} = \sum_k [A(x_{ii}^{(k)}), A(y_{ii}^{(k)})]|_{\mathfrak{A}_{ii}}$. We put $T_{ii} := \sum_k [A(x_{ii}^{(k)}), A(y_{ii}^{(k)})]$ and verify easily $T - \sum_{i=1}^{q} T_{ii} \in \mathfrak{a}_R$. As $T_{ii} \in \mathfrak{a}_{ii}$ the theorem is proved.

Now we are going to describe Lie $\mathrm{Aut}(K^o, \tilde{\gamma})$. We remark first that, by [4, Lemmata 3.1, 2.3], the Lie-algebras $\mathfrak{g}_{ii}$ are reductive and so are the sums of $[\mathfrak{g}_{ii}, \mathfrak{g}_{ii}] =: \mathfrak{h}_{ii}$ and their respective centers $\mathfrak{z}_{ii}$. Further, $\mathfrak{a}_R$ is a compact Lie-algebra, and so it is reductive [2, Chapitre IV, §4].

We write $\mathfrak{a}_R = \mathfrak{a}_\mathfrak{z} + \mathfrak{a}_\mathfrak{h}$ where $\mathfrak{a}_\mathfrak{z}$ is the center of $\mathfrak{a}_R$ and $\mathfrak{a}_\mathfrak{h} := [\mathfrak{a}_R, \mathfrak{a}_R]$. Summarizing, we obtain

$$\mathfrak{g}_{ii} = \mathfrak{z}_{ii} + \mathfrak{h}_{ii}, \quad \mathfrak{h}_{ii} = [\mathfrak{g}_{ii}, \mathfrak{g}_{ii}], \quad 1 < i < q. \tag{7.3}$$

$$\mathfrak{a}_R = \mathfrak{a}_{\mathfrak{z}} + \mathfrak{a}_{\mathfrak{h}}, \quad \mathfrak{a}_{\mathfrak{h}} = [\mathfrak{a}_R, \mathfrak{a}_R]. \tag{7.4}$$

$$\mathfrak{n} := \bigoplus_{1 < i < j < q} \mathfrak{g}_{ij} \quad \text{(sum of vector spaces)}. \tag{7.5}$$

THEOREM 7.3. *Let $M$ be a $J$-morphism of homogeneous cones. Then*

(a) Lie $\mathrm{Aut}(K^\sigma, \tilde{\gamma}) = \mathfrak{n} + \mathfrak{a}_R + \bigoplus_{i=1}^q \mathfrak{g}_{ii}$ *(direct sum of vector spaces).*

(b) $\mathfrak{a}_{\mathfrak{z}} \oplus \bigoplus_{i=1}^q \mathfrak{z}_{ii}$ *is an abelian, algebraic Lie-algebra of semisimple endomorphisms (direct sum of Lie-algebras).*

(c) $\mathfrak{a}_{\mathfrak{h}} \oplus \bigoplus_{i=1}^q \mathfrak{h}_{ii}$ *is a maximal semisimple Lie-subalgebra of* Lie $\mathrm{Aut}(K^\sigma, \tilde{\gamma})$ *(direct sum of Lie-algebras).*

(d) $\mathfrak{n} + \mathfrak{a}_{\mathfrak{z}} + \bigoplus_{i=1}^q \mathfrak{z}_{ii}$ *is the radical of* Lie $\mathrm{Aut}(K^\sigma, \tilde{\gamma})$ *(direct sum of vector spaces).*

(e) $\mathfrak{n}$ *is the maximal ideal of* Lie $\mathrm{Aut}(K^\sigma, \tilde{\gamma})$ *which consists of nilpotent endomorphisms.*

PROOF. (a) Let $T \in$ Lie $\mathrm{Aut}(K^\sigma, \tilde{\gamma})$; then there exists $S \in \Sigma_{1<i<j<q} \mathfrak{g}_{ij}$ such that $Te = Se$, as is clear from the definition of the spaces $\mathfrak{g}_{ij}$. But now $T - S \in$ Lie $\mathrm{Aut}_e(K^\sigma, \tilde{\gamma})$ and we have Lie $\mathrm{Aut}(K^\sigma, \tilde{\gamma}) = \mathfrak{n} + \mathfrak{a}_R + \Sigma_{i=1}^q \mathfrak{g}_{ii}$ by Theorem 7.2. To show directness let $0 = N + A_R + \Sigma_{i=1}^q T_{ii}$, $N \in \mathfrak{n}$, $A_R \in \mathfrak{a}_R$, $T_{ii} \in \mathfrak{g}_{ii}$; we obtain $0 = Nx_{jj} + \Sigma_{i=1}^q T_{ii}x_{jj}$ for $1 < j < q$, $x_{jj} \in \mathfrak{A}_{jj}$. From this we conclude $N = 0$ and $T_{ii}\mathfrak{X}_{G(M)} = 0$. By [4, Lemmata 3.1, 2.3], we get $T_{ii} = 0$, and we are done.

To prove the other assertions we first remark that the algebra in part (c) is semisimple. Further, the algebras in parts (b) and (c) commute. Now, from [4, Satz 3.3], it follows that the algebra in part (d) (which we call $\mathfrak{r}$) is an ideal of Lie $\mathrm{Aut}(K^\sigma, \tilde{\gamma})$ and $\mathfrak{n}$ is an ideal of Lie $\mathrm{Aut}(K^\sigma, \tilde{\gamma})$. Obviously, $\mathfrak{n}$ consists of nilpotent elements and $[\mathfrak{r}, \mathfrak{r}] \subset \mathfrak{n}$. So $\mathfrak{r}$ is a solvable ideal of Lie $\mathrm{Aut}(K^\sigma, \tilde{\gamma})$. Using (a), (c) and (d) are now proved. Finally, note that the algebras $\mathfrak{g}_{ii}$ and $\mathfrak{a}_R$ are selfadjoint; hence $\mathfrak{z}_{ii}$ and $\mathfrak{a}_{\mathfrak{z}}$ consist of semisimple endomorphisms. So by [2, Chapitre V, §4.2], and the fact that Lie $\mathrm{Aut}(K^\sigma, \tilde{\gamma})$ is algebraic, (see §2) we obtain the remaining statements.

REMARK 7.4. In [15], È. Vinberg proved a similar decomposition of Lie $\mathrm{Aut}(K, \gamma)$ in the case where the mapping $\varphi$ is trivial (i.e., $\varphi = 0$, $\hat{V} = 0$, $\gamma = \iota(K; )$).

## 8. Algebraic characterization of homogeneous cones.

We are now prepared to prove some results on $\mathcal{C}_M$ that will help to obtain an algebraic characterization of $J$-morphisms of homogeneous cones.

In this section we often use the spaces $\mathfrak{g}_{ij}^*$ which have been defined in §7. We further deal with the Lie-algebra Lie Str $\hat{\mathfrak{A}}$ of the structure group of the Jordan-algebra $\hat{\mathfrak{A}}$. For the definition of the *structure group* of a Jordan-alge-

bra we refer to [1]. We note that in the case of a semisimple Jordan-algebra $\mathfrak{J}$ the Lie-algebra Lie Str $\mathfrak{J}$ is generated by the left multiplications of $\mathfrak{J}$ (see [1, IX, Satz 3.1 and Satz 5.1]).

LEMMA 8.1. *Let $M = (F, \varphi, \hat{F})$ be a J-morphism of homogeneous cones and $\mathfrak{g}^* = \mathfrak{g}^{(\mathcal{C}_M)^*}$. Put $\tilde{\varphi}\colon \mathfrak{g}^* \to$ Lie Str $\hat{\mathfrak{A}}$, $\tilde{\varphi}(A_{ss}(x_{sj})) := \hat{A}_{\varphi(d_{ss})}(\varphi(x_{sj}))$ for $1 \leqslant j \leqslant s \leqslant q$, $x_{sj} \in \mathfrak{A}_{sj}$. Then we have*
  (a) *$\tilde{\varphi}$ induces a homomorphism of Lie-algebras.*
  (b) *$\varphi(T^\sigma x) = \tilde{\varphi}(T^\sigma)Q(x)$ for all $x \in V$, $T \in \mathfrak{g}$.*

PROOF. We have first to show that $\tilde{\varphi}$ is well defined. It obviously suffices to look at $\mathfrak{g}_{ii}^*$. But here Theorem 5.4(a), together with [4, Lemma 2.3], proves that $\tilde{\varphi}$ induces a homomorphism of Lie-algebras from $\mathfrak{g}_{ii}^*$ to Lie Str $\hat{\mathfrak{A}}$ (which we denote again by $\tilde{\varphi}$). Let now $T, S \in \mathfrak{n} = \bigoplus_{1 \leqslant i < j \leqslant q} \mathfrak{g}_{ij}$ and $R \in \bigoplus_{i=1}^{q} \mathfrak{g}_{ii}$. To prove (a) we have to show

$$\tilde{\varphi}([T^\sigma, S^\sigma]) = [\tilde{\varphi}(T^\sigma), \tilde{\varphi}(S^\sigma)] \quad \text{and} \quad \tilde{\varphi}([R^\sigma, T^\sigma]) = [\varphi(R^\sigma), \varphi(T^\sigma)].$$

To verify the first identity we choose a $\Phi$ as in Corollary 2.4. Then by Lemma 6.2 we have $T^\sigma, S^\sigma \in$ Lie $\dot{\Phi}$ and $(T^\sigma, \tilde{\varphi}(T^\sigma)) \in$ Lie $\Phi$ by Theorem 3.1. So the elements of $\mathfrak{n}$ satisfy (b). Further, from Theorem 3.1 and Lemma 2.5 the first identity follows. To get the second identity, first note that $\hat{\mathfrak{A}}$ is a Jordan-algebra. This implies, by [5, Theorem 4.6], that $\hat{\mathfrak{A}}$ and $\hat{\mathfrak{S}} = \mathfrak{S}_{\hat{F}}$ are equal. Hence $\hat{A}(\hat{u}) \in$ Lie Aut$(\hat{K}, \hat{\eta})$, $\hat{u} \in \hat{\mathfrak{A}}$. From Theorem 5.5 we conclude $(A(x_{ii}), \hat{A}(\varphi(x_{ii}))) \in$ Lie $\Pi_M$ (and (b) is proved). We form the Peirce-decomposition $\hat{\mathfrak{A}} = \bigoplus_{1 \leqslant i < j \leqslant q} \hat{\mathfrak{A}}_{ij}$ of $\hat{\mathfrak{A}}$ with respect to $\varphi(e_{11}), \ldots, \varphi(e_{qq})$. This is a $q$-$\mathfrak{R}$-decomposition of (the Jordan-algebra) $\hat{\mathfrak{A}}$ as is easily checked using [5, Lemma 1.4(b)]. From the remarks above, we have $(T^\sigma, \tilde{\varphi}(T^\sigma))$, $(R^\sigma, \tilde{\varphi}(R^\sigma)) \in$ Lie $\Pi_M$; hence $([R^\sigma, T^\sigma], [\tilde{\varphi}(R^\sigma), \tilde{\varphi}(T^\sigma)]) \in$ Lie $\Pi_M$. From Theorem 7.3 we see $[R^\sigma, T^\sigma] \in \mathfrak{n}^\sigma$ and $L_1 := [\varphi(R^\sigma), \tilde{\varphi}(T^\sigma)] \in \hat{\mathfrak{n}}^{\hat{\sigma}}$ where $\hat{\mathfrak{n}}$ is defined for the $q$-$\mathfrak{R}$-decomposition of $\hat{\mathfrak{A}}$. On the other hand, we have $([R^\sigma, T^\sigma], \tilde{\varphi}([R^\sigma, T^\sigma]))$ $\in$ Lie $\Pi_M$ with $L_2 := \tilde{\varphi}([R^\sigma, T^\sigma]) \in \hat{\mathfrak{n}}^{\hat{\sigma}}$. Therefore $L_1 - L_2$ is nilpotent and $0 = \varphi(0e) = (L_1 - L_2)\hat{e}$. But from [5, (1.13)], we get–using the triple $\hat{F}$–that $L_1 - L_2$ is skew adjoint with respect to $\hat{\sigma}$ and we are done.

We are going to characterize the space $\mathfrak{A}_{ii}$ of the $q$-$\mathfrak{R}$-decomposition $\mathcal{C}_M$ by some conditions which involve the cones $Y^{(s)}$ which are formed with respect to $\mathcal{C}_M$. The construction of $Y^{(s)}$ was first carried out in [4, §4]. Let $\mathcal{C}$ be a $q$-$\mathfrak{R}$-decomposition of the algebra $\mathfrak{C}$. We use the notations of §6 and define inductively homogeneous cones $Y^{(s)}$ for the algebras $\mathfrak{C}^{(s)}$. We start with $Y^{(q)}$, the domain of positivity associated to the formally-real Jordan-algebra $\mathfrak{C}_{qq}$ (see [9, VI, §5]). Assume that $Y^{(s)}$ is defined for $1 \leqslant k < s \leqslant q$, and denote by $Y_{kk}$ the domain of positivity associated to the formally-real Jordan-algebra $\mathfrak{C}_{kk}$. Then we split $\mathfrak{C}^{(k)} := \mathfrak{C}_1^{(k)} + \mathfrak{C}_{1/2}^{(k)} + \mathfrak{C}_0^{(k)}$ into the sum of the spaces

$\mathfrak{C}_1^{(k)} := \mathfrak{C}_{kk}$, $\mathfrak{C}_{1/2}^{(k)} := \bigoplus_{s-k+1}^q \mathfrak{C}_{ks}$, $\mathfrak{C}_0^{(k)} := \mathfrak{C}^{(k+1)}$ and put $Y^{(k)} := \{x_1 + x_{1/2} + x_0 \in \mathfrak{C}^{(k)},\ x_i \in \mathfrak{C}^{(k)},\ x_1 \in Y_{kk},\ x_0 - \frac{1}{2}d(x_{1/2}, R(x_1)^{-1}x_{1/2}) \in Y^{(k+1)}\}$ where $R(x_1)$ denotes the restriction of $C(x_1)$ to $\mathfrak{C}_{1/2}^{(k)}$, and $d$ equals $\Sigma_{s-k+1}^l c_{ss}$. We put $Y_{\mathcal{C}} := Y^{(1)}$. For $\mathcal{C} = \mathcal{Q}_M$ we also abbreviate $Y_{\mathcal{Q}_M}$ by $Y_M$.

We are ready to characterize $\mathfrak{A}_{ii}$.

In the setting of Lemma 8.1, we define $\mathfrak{N}_s$, $1 \leqslant s \leqslant q$, to be the set of $y \in \mathfrak{A}^{(s)}$, such that

(8.1) $A_y(u)$ is a derivation of $\mathfrak{A}_{e_{kk}}^{(k)}$ for all $u \in \mathfrak{A}^{(s)}$ and $1 \leqslant k < s \leqslant q$;

(8.2) for all $u \in \mathfrak{A}^{(s)}$ there exists a $\hat{T} \in \text{Lie Aut}(\hat{K}, \hat{\eta})$ such that for all $x \in V$ we have $\varphi([A_y(u)]^\sigma x) = \hat{T}\varphi(x)$.

(8.3) $[\exp tA_y(u)]Y^{(s)} = Y^{(s)}$ for all $u \in \mathfrak{A}^{(s)}$, $t \in \mathbf{R}$ (with $Y^{(s)}$ constructed for $\mathcal{Q}_M$ as above).

LEMMA 8.2. $\mathfrak{N}_s = \mathfrak{A}_{ss}$.

PROOF. First we prove "$\supset$". Here (8.1) is clear from Theorem 6.3 and the definition of a $q$-$\mathfrak{N}$-decomposition. To prove (8.2), we first note that $(A_{e_{ss}})_{x_{ss}}(x_{sj}) = A_{x_{ss}}(x_{sj})$ by [4, Lemma 2.6]. Then we use Lemma 8.1 and obtain (8.2). Finally, the identity (8.3) follows from Lemma 6.3 and [4, Satz 4.2]. The converse inclusion we prove by induction on $q$. For $q = 1$ there is nothing to show. Now let $q > 1$ and $1 \leqslant s \leqslant q$. If $s = 1$, then (8.2) and (8.3) imply $[A_y(u)]^\sigma \in \text{Lie Aut}(K, \gamma)$ and so $y \in \mathfrak{S}_{\widetilde{G(M)}} = \mathfrak{A}_{11}$. Assume now that $s \geqslant 2$. Here from (8.1) we derive

$$A_{1/2}([A_y(u)]^\sigma v_0) = \{[A_y(u)]^\sigma A_{1/2}(v_0) + A_{1/2}(v_0)[A_y(u)]\}|_{\mathfrak{A}_{1/2}}$$

for all $v_0 \in \mathfrak{A}_0(\tilde{c})$. A comparison with (4.7) shows that $A_y(u)$ satisfies the conditions (8.1), (8.2) and (8.3) with respect to $M_0$ and its derived $(q-1)$-$\mathfrak{N}$-decomposition $\mathcal{Q}_{M_0}$. By induction now $y \in \mathfrak{A}_{ss}$ follows.

The last two lemmata give a hint what conditions on a $q$-$\mathfrak{N}$-decomposition $\mathcal{C}$ we have to put so that $\mathcal{C}$ is of type $\mathcal{Q}_M$. As our goal is to characterize those $J$-morphisms of homogeneous cones where $\varphi$ starts from a fixed vector space $V$ and induces a fixed linear form $\hat{x} \mapsto \hat{\sigma}(\hat{e}, \hat{x})$ on a fixed Jordan-algebra $\mathfrak{J} = \hat{\mathfrak{A}}$, we first make some remarks on Jordan-algebras and associative linear forms and then give the final definitions and results.

Let $\mathfrak{J}$ be a formally-real Jordan-algebra. We denote the unit of $\mathfrak{J}$ by $e_{\mathfrak{J}}$ and the left multiplications in $\mathfrak{J}$ by $J(x)$. We put $xy = J(x)y$. Further, Pos $\mathfrak{J}$ is defined to be the domain of positivity associated to $\mathfrak{J}$, Pos $\mathfrak{J} := \{x^2;\ x$ invertible in $\mathfrak{J}\}$. For details on the cone Pos $\mathfrak{J}$ we refer to [1, XI] or [9].

Now let $\lambda: \mathfrak{J} \to \mathbf{R}$ be an associative, positive-definite linearform on $\mathfrak{J}$ [1, XI]. Splitting $\mathfrak{J}$ into a sum of simple ideals $\mathfrak{J}^{(i)}$, we see by [1, I, Satz 6.4], and [1, XI, §3], that the restriction of $\lambda$ to $\mathfrak{J}^{(i)}$ is a multiple of the linearform $x_i \mapsto \text{trace } J(x_i)$, $x_i \in \mathfrak{J}^{(i)}$. Say $\lambda(x_i) = l_i \text{ trace } J(x_i)$. We put

$$\eta_{\mathfrak{J}}^{\lambda}(x) := \prod_{i} \iota\left(P_{\mathfrak{J}^{(i)}}; x_i\right)^{l_i}$$

and easily get

$$\lambda(u) = -\Delta_x^u \log \eta_{\mathfrak{J}}^{\lambda}(x)\big|_{x=e_{\mathfrak{J}}}.$$

After these preparations we can give the following definitions.

*Let $V$ be a finite-dimensional vector space over $\mathbf{R}$ and $\mathfrak{J}$ a formally-real Jordan-algebra. Let further $\lambda: \mathfrak{X} \to \mathbf{R}$ be an associative, positive-definite linear form on $\mathfrak{J}$.*

*By $\mathfrak{M}(V, \mathfrak{J}, \lambda)$ we denote the set of J-morphisms $M = (F, \varphi, \hat{F})$ of homogeneous cones such that*

$$V_F = V \quad \text{and} \quad \eta_F = \iota(K_F; ), \qquad \eta_F(e_F) = 1, \tag{8.4.a}$$

$$\mathfrak{A}_{\hat{F}} = \mathfrak{J} \quad \text{and} \quad \hat{\eta} = \eta_{\mathfrak{J}}^{\lambda}. \tag{8.4.b}$$

*Further, we denote by $\mathfrak{K}(V, \mathfrak{J}, \lambda)$ the set of triples $(\mathfrak{C}, \mathcal{C}, \varphi)$ such that*

(8.5.a) $\mathfrak{C}$ *is a commutative algebra on $V$ (with unit $e_{\mathfrak{C}}$ and left multiplications $C(v)$),*

(8.5.b) $\tau(e_{\mathfrak{C}}, v) = \text{trace } C(v) + \lambda(\varphi(v))$ *is an associative, positive-definite linear form of $\mathfrak{C}$,*

(8.5.c) $\mathcal{C}$ *is a q-$\mathfrak{K}$-decomposition (for some q),*

(8.5.d) $\varphi: \mathfrak{C} \to \text{Lie Str } \mathfrak{J}$ *is a linear mapping such that*

(i) $\varphi(e_{\mathfrak{C}}) = e_{\mathfrak{J}}$;

(ii) *the mapping $\tilde{\varphi}: \mathfrak{g}^* \to \text{Lie Str } \mathfrak{J}$, defined by*

$$\tilde{\varphi}\left(C_{ss}(x_{sj})\right) := J_{\varphi(c_{ss})}\left(\varphi(x_{sj})\right), \qquad 1 < j < s < q, \ x_{sj} \in \mathfrak{C}_{sj},$$

*induces a homomorphism of Lie-algebras,*

(iii) $\varphi(T^{\tau}x) = \tilde{\varphi}(T^{\tau})\varphi(x)$ *for all $T \in \mathfrak{g}^{\mathfrak{C}}$, $x \in \mathfrak{C}$.*

(8.5.e) *The set of $y \in \mathfrak{C}^{(s)}$, $1 < s < q$, such that*

(j) $C_y(u)$ *is a derivation of $\mathfrak{C}_{c_{kk}}^{(k)}$ for all $u \in \mathfrak{C}^{(s)}$ and all $1 < k < s < q$;*

(jj) *for all $u \in \mathfrak{C}^{(s)}$ there exists a $\hat{T} \in \text{Lie Str } \mathfrak{J}$ such that for all $x \in V$, $\varphi([C_y(u)]^{\tau}x) = \hat{T}\varphi(x)$;*

(jjj) $[\exp tC_y(u)]Y^{(s)} = Y^{(s)}$ *for all $u \in \mathfrak{C}^{(s)}$, $t \in \mathbf{R}$, is equal to $\mathfrak{C}_{ss}$.*

We shall show that there exists a canonical bijection of $\mathfrak{M}(V, \mathfrak{J}, \lambda)$ onto $\mathfrak{K}(V, \mathfrak{J}, \lambda)$. The first step is to show that the mappings (8.6) and (8.7) are well defined.

$$\mathfrak{H}_1: \mathfrak{M}(V, \mathfrak{J}, \lambda) \to \mathfrak{K}(V, \mathfrak{J}, \lambda), \qquad M \mapsto (\mathfrak{A}_{G(M)}, \mathfrak{C}_M, \varphi). \tag{8.6}$$

LEMMA 8.3. $\mathfrak{H}_1$ *is well defined.*

PROOF. We have to check the various conditions (8.5.∗). But (8.5.a) and (8.5.d)(i) are clear. The identity (8.5.b) is easily verified. The remaining conditions follow by Theorem 6.3 and Lemmata 8.1, 8.2.

$$\mathfrak{H}_2\colon \mathfrak{R}(V, \mathfrak{I}, \lambda) \to \mathfrak{M}(V, \mathfrak{I}, \lambda), \qquad (\mathfrak{L}, \mathcal{C}, \varphi) \mapsto M = (F, \varphi, \hat{F}) \quad (8.7)$$

with $F = \langle Y_{\mathcal{C}}^{\tau}, \iota(Y_{\mathcal{C}}^{\tau};\,), e_{\mathfrak{C}} \rangle$ and $\hat{F} = \langle \mathrm{Pos}\,\mathfrak{I}, \eta_{\mathfrak{I}}^{\lambda}, e_{\mathfrak{I}} \rangle$ where $\iota(Y_{\mathcal{C}}^{\tau}; e_{\mathfrak{C}}) = 1$.

LEMMA 8.4. $\mathfrak{H}_2$ is well defined and for all $(\mathfrak{C}, \mathcal{C}, \varphi) \in \mathfrak{R}(V, \mathfrak{I}, \lambda)$ there exists a group $\Gamma$ which satisfies (M.4) for $\mathfrak{H}_2((\mathfrak{C}, \mathcal{C}, \varphi))$ and for which $\mathrm{Lie}\,\Gamma \subset \mathfrak{g}^{\mathcal{C}*}$ holds.

PROOF. Obviously we just have to prove that $M$ is a morphism of homogeneous cones. Now (M.1) and (M.2) are clear and from (8.5.d)(iii), we conclude $\varphi([\exp tT^{\tau}]x) = [\exp t\tilde{\varphi}(T^{\tau})]\varphi(x)$ for all $T \in \mathfrak{g}^{\mathcal{C}}$. As $\tilde{\varphi}(T^{\tau}) \in$ Lie Str $\mathfrak{I}$ we have $\exp t\tilde{\varphi}(T^{\tau}) \in \mathrm{Aut}(\mathrm{Pos}\,\mathfrak{I}, \eta_{\mathfrak{I}}^{\lambda})$ (note that Lie $\mathrm{Aut}(\mathrm{Pos}\,\mathfrak{I}, \eta_{\mathfrak{I}}^{\lambda})$ = Lie Str $\mathfrak{I}$ as can be derived from [1, XI, Satz 4.6]). Since $\varphi(e_{\mathfrak{C}}) = e_{\mathfrak{I}} \in$ Pos $\mathfrak{I}$, (M.3) follows. Let further $\mathfrak{t} = \mathfrak{n} + \bigoplus \mathfrak{t}_{ii}$, where $\mathfrak{t}_{ii}$ is a trigonalizable subalgebra of $\mathfrak{g}_{ii}^{\mathcal{C}}$ generating a transitive group of the domain of positivity of $\mathfrak{C}_{ii}$. Then the group $\Gamma$ generated by $\mathfrak{t}^{\tau}$ is a closed subgroup of $\mathrm{Aut}(Y_{\mathcal{C}}^{\tau}, \iota(Y_{\mathcal{C}}^{\tau};\,))$ satisfying the conditions of (M.4). Obviously, Lie $\Gamma \subset \mathfrak{g}^{*}$.

THEOREM 8.5. $\mathfrak{H}_1 \cdot \mathfrak{H}_2 = \mathrm{Id}$ and $\mathfrak{H}_2 \cdot \mathfrak{H}_1 = \mathrm{Id}$.

PROOF. Let $M = (F, \varphi, \hat{F}) \in \mathfrak{M}(V, \mathfrak{I}, \lambda)$. We compare the construction of $Y_M$ as given in [4, §4, 2] (it has been recalled at the beginning of this section) with [5, Theorem 1.8] (applied to $G(M)$ and $\tilde{c}$) and see that $Y_M = K^{\sigma}$, $\sigma = \sigma_{G(M)}$, holds. From this $\mathfrak{H}_2 \circ \mathfrak{H}_1 = \mathrm{Id}$ easily follows. Let $(\mathfrak{C}, \mathcal{C}, \varphi) \in \mathfrak{R}(V, \mathfrak{I}, \lambda)$. With $M = (F, \varphi, \hat{F})$ as in (8.7) we have to show: $\mathfrak{A}_{G(M)} = \mathfrak{C}$ and $\mathcal{C} = \mathcal{C}_M$. First we calculate $\sigma = \sigma_{G(M)}$. We have

$$\gamma_M(x) = \gamma(x) = \iota(Y_{\mathcal{C}}^{\tau}, x)\eta_{\mathfrak{I}}^{\lambda}(\varphi(x))$$

and put

$$\tau(h_{\gamma,\tau}(x), u) := -\Delta_x^u \log \gamma(x), \qquad x \in Y_{\mathcal{C}}^{\tau}. \quad (1)$$

From this we conclude

$$h_{\gamma,\tau}(W^{\tau}x) = W^{-1}h_{\gamma,\tau}(x); \quad \text{for all } x \in Y_{\mathcal{C}}^{\tau}, W \in \mathrm{Aut}(Y_{\mathcal{C}}, \iota(Y_{\mathcal{C}};\,)). \quad (2)$$

By $-\Delta_x^u\iota(Y^{\tau}; x)|_{x=e_{\mathfrak{C}}} = \mathrm{trace}\,T$ for $T \in \mathrm{Lie}\,\mathrm{Aut}(Y_{\mathcal{C}}^{\tau}, \iota(Y_{\mathcal{C}}^{\tau};\,))$ satisfying $Te_{\mathfrak{C}} = u$ and [4, Folgerung 3.4], we get further

$$\tau(h_{\gamma,\tau}(e_{\mathfrak{C}}), u) = \mathrm{trace}\,C(u) + \lambda(\varphi(u)), \qquad u \in V. \quad (3)$$

Comparing the right-hand side with (8.5.b) we see that

$$h_{\gamma,\tau}(e_{\mathfrak{C}}) = e_{\mathfrak{C}}. \quad (4)$$

We choose a $\Gamma$ which satisfies the conditions of (M.4) and Lie $\Gamma \subset \mathfrak{g}^{\mathcal{C}*} = \mathfrak{g}^{*}$. This is possible by Lemma 8.4. We use Satz 6.2 and Lemma 5.4 of [4] and get a diffeomorphism $j\colon Y_{\mathcal{C}} \to Y_{\mathcal{C}}^{\tau}$ which has the properties $j^{-1}(e_{\mathfrak{C}}) = e_{\mathfrak{C}}$ and $j^{-1}(W^{\tau}x) = W^{-1}j^{-1}(x)$ for all $x \in Y_{\mathcal{C}}^{\tau}, W^{\tau} \in \Gamma$.

From this a comparison with (4) and (2) gives

$$h_{\gamma, \tau} = j^{-1}. \tag{5}$$

By [4, Lemma 5.5], we conclude $\sigma(u, v) = \Delta_x^u \Delta_x^v \log \gamma(x)|_{x = e_{\mathfrak{C}}} = -\Delta_x^u \tau(j^{-1}(x), v) = \tau(u, v)$. We apply [7b, (1.17)] and [4, Folgerung 3.4], to Lie $\Gamma \subset \mathfrak{g}^*$ and get $2C(Te) = T + T^\tau = T + T^\sigma = 2A(Te)$ for $T \in$ Lie $\Gamma$ (here $A(u)$ denotes as usual the left multiplication of the algebra $\mathfrak{A}_{G(M)}$). Hence we have proved $\mathfrak{A}_{G(M)} = \mathfrak{C}$. To see $\mathcal{C} = \mathcal{C}_M$ it is enough to show $c_{ii} = e_{ii}$. From (8.5.e) we obtain $c_{11} = e_{11}$.

Now we consider the triple $(\mathfrak{C}_0, \mathcal{C}_0, \varphi_0)$ where $\mathfrak{C}_0 = \mathfrak{C}_0^{(1)} = \mathfrak{C}^{(2)}$, $\mathcal{C}_0 := (\mathfrak{C}_{ij}; 2 \leqslant i, j \leqslant q)$ and $\varphi_0 := \varphi|_{\mathfrak{C}_0}$. Further, we define $\varphi_0' : \mathfrak{C}_0 \to \text{Sym}(\mathfrak{C}_{1/2}^{(1)}, \tau) \times \mathfrak{J}_0 := \mathfrak{J}_0'$, $v_0 \mapsto (C_{1/2}(v_0), \varphi(v_0))$ where $\mathfrak{J}_0 := \mathfrak{J}_0(\varphi(c_{11}))$ and $C_{1/2}(v_0) := C(v_0)|_{\mathfrak{C}_{1/2}^{(1)}}$. Finally, we put $\lambda_0 : \mathfrak{J}_0' \to \mathbf{R}$, $\lambda_0(B, \hat{x}) := \text{trace } B + \lambda(\hat{x})$. The Jordan-structure in $\text{Sym}(\mathfrak{C}_{1/2}^{\mathcal{C}}, \tau)$ is defined by $(A, B) \mapsto AB + BA$. Hence the unit of $\mathfrak{J}_0'$ is $e' = e_{\mathfrak{J}_0} = (\frac{1}{2} \text{ Id}, e_{\mathfrak{J}_0})$. We claim $(\mathfrak{C}_0, \mathcal{C}_0, \varphi_0') \in \mathfrak{R}(V_0, \mathfrak{J}_0', \lambda_0)$, where $V_0 = \mathfrak{C}_0$ as a vector space. We have to verify (8.5.*) and see that the first three conditions are satisfied; further (8.5.d)(i) is clear.

Now let $T \in \mathfrak{g}^{\mathcal{C}_0}$. Then from the definition of a $q$-$\mathfrak{R}$-decomposition we get that $T$ is a derivation of $\mathfrak{C}_{c_{11}}$. From this we derive $C_{1/2}(T^\tau v_0) = T^\tau C_{1/2}(v_0) + C_{1/2}(v_0)T$ for all $v_0 \in \mathfrak{C}_0$. Now (8.5.d) is easily verified. Finally, let $y \in \mathfrak{C}^{(s)}$, $s > 2$, satisfy the conditions of (8.5.e); then from (jj) we obtain $C_{1/2}([C_y(u)]^\tau v_0) = T^\tau C_{1/2}(v_0) + C_{1/2}(v_0)T$ for some $T \in \text{End } \mathfrak{C}_{1/2}^{(1)}$ (it is well known that the elements of Lie Str $\text{Sym}(\mathfrak{C}_{1/2}^{(1)}, \tau)$ are of this type). Now [4, Lemma 2.4], shows that $C_y(u)$ is a derivation for $\mathfrak{C}_{c_{11}}^{(1)}$ and by the definition of $(\mathfrak{C}, \mathcal{C}, \varphi)$ we have $y \in \mathfrak{C}_{ss}$. On the other hand, since $(\mathfrak{C}, \mathcal{C}, \varphi) \in \mathfrak{R}(V, \mathfrak{J}, \lambda)$, all $y \in \mathfrak{C}_{ss}$ satisfy the conditions of (8.5.e). So we have $(\mathfrak{C}_0, \mathcal{C}_0, \varphi_0') \in \mathfrak{R}(V_0, \mathfrak{J}_0', \lambda_0)$. We apply $\mathfrak{H}_2$ to the triple $(\mathfrak{C}_0, \mathcal{C}_0, \varphi_0')$ and get a triple $M' = (F', \varphi_0', \hat{F}')$ with $F' = \langle Y_{\mathfrak{C}_0}^\tau, \iota(Y_{\mathfrak{C}_0}^\tau; ), e_{\mathfrak{C}_0} \rangle$ and $\hat{F}' = \langle \text{Pos } \mathfrak{J}_0', \eta_{\mathfrak{J}_0}^{\lambda_0}, e_{\mathfrak{J}_0} \rangle$. A comparison with the definitions of §4 gives $M_0 = M'$ where $M = \mathfrak{H}_2(\mathfrak{C}, \mathcal{C}, \varphi)$ as above. By induction hypothesis we have $\mathcal{C}_{M_0} = \mathcal{C}_0$ and from the definition of $\mathcal{C}_M$ we derive easily $\mathcal{C}_{M_0} = (\mathcal{C}_M)_0$. We get $c_{ii} = e_{ii}$ for $2 \leqslant i \leqslant q$. We recall that $c_{11} = e_{11}$ was already shown above and the theorem is proved.

Finally, we prove the theorem announced in [4]. For this we specialize Theorem 8.5 to the case where "$\mathfrak{J} = 0$"; this amounts to just saying that $\mathfrak{J}$ and $\varphi$ do not appear. We reformulate the conditions (8.5.*) for this case.

A $q$-$\mathfrak{R}$-decomposition $\mathcal{C}$ is called *optimal* iff

(8.8.a) $\tau(e_{\mathfrak{C}}, v) = \text{trace } C(v)$ is an associative, positive-definite linearform of $\mathfrak{C}$;

(8.8.b) the set of $y \in \mathfrak{C}^{(s)}$, $1 \leqslant s \leqslant q$, such that

(j) $C_y(u)$ is a derivation of $\mathfrak{C}_{c_{kk}}^{(k)}$ for all $u \in \mathfrak{C}^{(s)}$ and all $1 \leqslant k < s \leqslant q$.

(jj) $[\exp t C_y(u)] Y^{(s)} = Y^{(s)}$ for all $u \in \mathfrak{C}^{(s)}$, $t \in \mathbf{R}$, is equal to $\mathfrak{C}_{ss}$.

We have to remark that the definition of an optimal $q$-$\mathfrak{R}$-decomposition as given in [4] was not complete although the theorem was correctly stated there.

Above we have simplified the definition of $\mathfrak{R}(V, \mathfrak{J}, \lambda)$ under the additional assumption "$\mathfrak{J} = 0$".

As to the conditions (8.4.∗) we note that $\iota(K; )$ is uniquely determined by $K$ and $e$, so in the case where $\mathfrak{J}$ and $\varphi$ do not appear, we see that $\mathfrak{M}(V, \mathfrak{J}, \lambda)$ is equal to the set of pairs $(K, e)$ where $K$ is a homogeneous cone and $e$ a point in $K$.

Summing up, we have

THEOREM 8.6. *For each finite-dimensional vector space $V$ over $\mathbf{R}$ there exists a canonical bijection of the set of pairs $(K, e)$, where $K$ is a homogeneous cone in $V$ and $e$ a point of $K$, onto the set of all optimal $q$-$\mathfrak{R}$-decompositions ($q$ arbitrary) of algebras on $V$.*

LIST OF NOTATIONS

| | |
|---|---|
| $K$ | regular cone |
| $\mathfrak{F}$ | class of triples considered |
| $\langle K, \eta, e \rangle$ | typical triple of $\mathfrak{F}$ |
| $K_D, \eta_D, e_D$ | components of a triple $D$ of $\mathfrak{F}$ |
| $\Delta^u_x f(x)$ | differentiation of $f$ at $x$ in direction $u$ |
| $\mathrm{Aut}(K, \eta)$ | group, leaving almost invariant $\eta$ |
| $V_D$ | vectorspace generated by $K_D$ |
| $\sigma_D$ | bilinearform associated to the triple $D$ of $\mathfrak{F}$ |
| $h_D$ | logarithmic gradient of $\eta_D$ |
| $H_D$ | negative differential of $h_D$ |
| $\mathfrak{A}_D$ | algebra associated to the triple $D$ of $\mathfrak{F}$ |
| $K^\sigma$ | $\sigma$-dual cone for $K$ with respect to $\sigma$ |
| $M = (F, \varphi, F)$ | morphism of homogeneous cones |
| $G(M)$ | triple of $\mathfrak{F}$ associated to $M$ |
| $\gamma_M$ | $\gamma_M = \eta_{G(M)}$ |
| $\Gamma(h)$ | algebraic group defined with respect to the rational map $h$ |
| $\Pi_M, \Pi^a_M$ | groups defined for $M$ |
| $\mathrm{Lie}\ \Gamma$ | Lie-algebra of the Lie-group $\Gamma$ |
| $\Gamma^0$ | connected component of the unit of $\Gamma$ |
| $\mathfrak{X}_D$ | Jordan-algebra associated to the triple $D$ of $\mathfrak{F}$ |
| CSI | complete system of orthogonal idempotents |
| $\mathfrak{A}_{ij}$ | Peirce-spaces |
| $\mathfrak{A}_\nu(c)$ | Peirce-spaces (with respect to the idempotent $c$) |
| ctt-group | connected, transitive, trigonalizable |
| ctt-algebra | Lie-algebra of a ctt-group |
| $\mathfrak{B}_b$ | mutant of the algebra $\mathfrak{B}$ with respect to $b$ |
| $B_b(x)$ | leftmultiplications in $\mathfrak{B}_b$ if $B(x)$ are the leftmultiplications of $\mathfrak{B}$ |
| $\mathfrak{t}_{ij}$ | summands of a ctt-algebra |
| $\iota(K; )$ | "invariant" of the regular cone $K$ |
| $K^*$ | dual cone for $K$ |
| $\tilde{\eta}$ | function on $K^\sigma$ |
| $\tilde{R}$ | dual triple for a triple $R$ of $\mathfrak{F}$ |

$T^\sigma$ — adjoint of the endomorphism $T$ with respect to $\sigma$

$\mathfrak{S}_R$ — set of "Jordan-structures" relative the triple $R$ of $\mathfrak{F}$

$c = c_{G(M)}$ — unit of $\mathfrak{S}_{G(M)}$

$\tilde{c} = c_{\widetilde{G(M)}}$ — unit of $\mathfrak{S}_{\widetilde{G(M)}}$

$K_i, \eta_i$

$P(\mathfrak{A}_{1/2}, \sigma)$ — cone of positive-definite endomorphisms

$\mathrm{Sym}(\mathfrak{A}_{1/2}, \sigma)$ — selfadjoint endomorphisms (relative $\sigma$)

$\varphi_i$

$r_i, R_i, \hat{R}_i, \sigma_i$

$M_i$

$J$-morphism

$x.yz$ — $x.yz = x(yz)$

$N^{(i)}, F^{(i)}, \varphi^{(i)}, F^{(i)}, G^{(i)}$

$e_{ii}$ — CSI associated to $\mathfrak{C}_M$

$q = q_M$ — length of the CSI associated to $\mathfrak{C}_M$

$\mathfrak{C}_M$ — Peirce-decomposition associated to $M$

$q$-$\mathfrak{R}$-decomposition

$\mathfrak{C}^{(k)}$

$C_{rr}^{(k)}(x)$ — leftmultiplications of the mutant of $\mathfrak{C}^{(k)}$ by $c_{rr}$

$\mathrm{Aut}_a(K, \eta)$ — isotropy group at $a$

$\mathfrak{g}_{ij}, \mathfrak{g}_{ij}^c$

$\mathfrak{g}, \mathfrak{g}^c$ — Lie-algebra associated to the $q$-$\mathfrak{R}$-decomposition

$\mathcal{C}^*$ — dual $q$-$\mathfrak{R}$-decomposition

$\mathfrak{g}_{ij}^*, \mathfrak{g}^*$

$\mathfrak{a}_{ii}, \mathfrak{a}_R$ — summands of the Lie-algebra of the isotropy group

$\mathfrak{h}_{ii}$ — semisimple part of $\mathfrak{g}_{ii}$

$\mathfrak{z}_{ii}$ — center of $\mathfrak{g}_{ii}$

$\mathfrak{a}_{\mathfrak{h}}$ — semisimple part of $\mathfrak{a}_R$

$\mathfrak{a}_{\mathfrak{z}}$ — center of $\mathfrak{a}_R$

$\mathfrak{n}$ — maximal ideal of Lie $\mathrm{Aut}(K^\sigma, \tilde{\gamma})$ consisting of nilpotent endomorphisms

$\mathrm{Lie\ Str}\ \mathfrak{J}$ — Lie-algebra of the structure group of $\mathfrak{J}$

$Y^{(s)}$ — cones associated to $\mathfrak{C}^{(s)}$

$Y_{ss}$ — domain of positivity associated to $\mathfrak{C}_{ss}$

$\mathfrak{C}_\nu^{(k)}$ — Peirce-spaces of $\mathfrak{C}^{(k)}$ with respect to $c_{kk}$

$Y_{\mathcal{C}}$ — $Y_{\mathcal{C}} = Y^{(1)}$

$Y_M$ — $Y_M = Y_{\mathfrak{C}_M}$

$\mathfrak{R}_s$

$\mathrm{Pos}\ \mathfrak{J}$ — domain of positivity associated to $\mathfrak{J}$

$\eta_{\mathfrak{J}}^\lambda$

$\mathfrak{M}(V, \mathfrak{J}, \lambda)$ — $J$-morphisms of homogeneous cones

$\mathfrak{R}(V, \mathfrak{J}, \lambda)$ — family of $q$-$\mathfrak{R}$-decompositions and mappings

$\mathfrak{H}_1$ — map from $\mathfrak{M}(V, \mathfrak{J}, \lambda)$ onto $\mathfrak{R}(V, \mathfrak{J}, \lambda)$

$\mathfrak{H}_2$ — map from $\mathfrak{R}(V, \mathfrak{J}, \lambda)$ onto $\mathfrak{M}(V, \mathfrak{J}, \lambda)$

optimal $q$-$\mathfrak{R}$-decomposition

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